

Box constrained ℓ_1 optimization in random linear systems – asymptotics

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Abstract

In this paper we consider box constrained adaptations of ℓ_1 optimization heuristic when applied for solving random linear systems. These are typically employed when on top of being sparse the systems' solutions are also known to be confined in a specific way to an interval on the real axis. Two particular ℓ_1 adaptations (to which we will refer as the *binary* ℓ_1 and *box* ℓ_1) will be discussed in great detail. Many of their properties will be addressed with a special emphasis on the so-called phase transitions (PT) phenomena and the large deviation principles (LDP). We will fully characterize these through two different mathematical approaches, the first one that is purely probabilistic in nature and the second one that connects to high-dimensional geometry. Of particular interest we will find that for many fairly hard mathematical problems a collection of pretty elegant characterizations of their final solutions will turn out to exist.

Index Terms: Phase transitions; large deviations; linear systems of equations; binary/box ℓ_1 .

1 Introduction

This paper provides a detailed mathematical study of specific properties of the well known ℓ_1 heuristic when used for solving linear systems of equations known to have solutions of particular form. These systems assume an $m \times n$ ($m \leq n$) system matrix A and an n dimensional vector $\tilde{\mathbf{x}}$ with real entries (for short we write $A \in \mathbb{R}^{m \times n}$ and $\tilde{\mathbf{x}} \in \mathbb{R}^n$). Then the standard matrix-vector multiplication of A and $\tilde{\mathbf{x}}$ gives

$$\mathbf{y} = A\tilde{\mathbf{x}}. \quad (1)$$

One is then interested in finding $\tilde{\mathbf{x}}$ if A and \mathbf{y} in (1) are given (clearly, by (1) such an $\tilde{\mathbf{x}}$ obviously exists). A particularly interesting variant of this problem that attracted a lot of attention over last several decades is the under-determined scenario with structured solutions. Namely, as is well known, in the under-determined scenario $m < n$ and if A is full rank (which will typically be assumed throughout the entire paper) the problem has multiple solutions and in many applications would not be among the best posed ones. However, through additional structuring of $\tilde{\mathbf{x}}$ one can make the above problem typically well posed (so that it actually has a unique solution). A type of structuring that has been of great interest for a long time assumes the so-called sparse solutions i.e. the sparse $\tilde{\mathbf{x}}$ and it is precisely in solving the linear systems known to have this type of solutions where the above mentioned ℓ_1 heuristic has been very successful. A heuristic type of explanation for this is the following simple line of arguments. One first recognizes that finding the sparsest $\tilde{\mathbf{x}}$ such that (1) holds amounts to solving

$$\begin{aligned} \min \quad & \|\mathbf{x}\|_0 \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{y}, \end{aligned} \quad (2)$$

where $\|\mathbf{x}\|_0$ is the so-called ℓ_0 (quasi) norm of \mathbf{x} that basically counts the number of nonzero entries of \mathbf{x} (of course, from this point on the assumption will always be that there is at least one \mathbf{x} that satisfies the constraints in (2), essentially $\tilde{\mathbf{x}}$ in (1)). (2) is of course well known to be notoriously hard to solve exactly.

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Nonetheless, one observes that $q = 1$ is the smallest q such that $\|\mathbf{x}\|_q = (\sum_{i=1}^n |\mathbf{x}_i|^q)^{\frac{1}{q}}$ is a convex function and relaxes (2) so that it becomes

$$\begin{aligned} \min \quad & \|\mathbf{x}\|_1 \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{y}. \end{aligned} \tag{3}$$

(3) is of course a much easier optimization problem than (2). In fact, not only is it a convex optimization problem due to the convexity of $\|\mathbf{x}\|_1$, it is actually a linear program relatively easily solvable in polynomial time (of course, there are many other heuristics/relaxations of (2) that one can alternatively employ see, e.g. [2, 6, 9, 12, 13, 27]; however our concern in this paper will be precisely the minimization of the ℓ_1 norm of \mathbf{x} from (3) and its variants that we will discuss below as they continue to stand, in our view, as an unbeatable benchmark when it comes to solving linear systems with sparsely structured solutions). Being a much easier optimization problem than (2) is, of course, a good feature of the ℓ_1 . However, on its own that would not be enough for its a massive use. Its excellent solving abilities and the existence of rigorous mathematical results that confirm such abilities contribute a great deal to the ℓ_1 's popularity as well. Moreover, while the practical applicability has been known for quite some time, the analytical progress flourished over the last decade. There has been a lot of great work in recent years about various aspects of the ℓ_1 . As the ℓ_1 in its core form (3) will not be the central point of this paper we leave a thorough discussion about its properties to review papers and here mention only the key milestones when it comes to its performance characterizations, namely [1, 8] where the initial, qualitative results were presented and [4, 5, 22, 23] where the ℓ_1 's exact performance characterizations were obtained. These, in our view, mathematically solidified the importance of (3) in studying linear inverse problems.

In this paper we will consider an upgrade to the standard sparse structuring mentioned above. Namely, we will be interested in unknown vectors that in addition to being sparse are also known to be from a given interval. When stated like this, one then recognizes that these kinds of vectors are not that much different from any vectors (simply one can always design an interval so that all components of any vector are from such an interval; obviously, we will throughout the paper consider so to say practically realistic scenarios, i.e. vectors that have finite components). To remove this ambiguity we will first introduce the so-called *binary* sparse vectors (later in the paper we will expand this definition so that it includes vectors that more faithfully resemble the ones with the elements from a given interval). Namely, the binary vectors will have each of their components equal either to zero or to one (more on this or similar discrete type of unknown vectors as well as on their potential applications can be found in e.g. [3, 6, 7, 10, 11, 25]). While it will be fairly obvious later on, we still take the opportunity right here at the beginning to emphasize that there is really nothing specific about zero and one and that instead of them one can choose basically any two real numbers and all of what we will present below will hold with minimal/trivial adjustments. Additionally, we will call binary vectors k sparse if they have k components equal to one and the remaining ones equal to zero. It is also relatively easy to note that the binary sparse vectors are a subclass of the so-called nonnegative sparse vectors studied in e.g. [7, 17, 23, 28]. One can of course still use the standard ℓ_1 to solve under-determined systems with nonnegative or binary sparse solutions. However, as it is by now well known (see, e.g. [4, 5, 22, 23]), a substantial performance improvement can be achieved if one slightly modifies the standard ℓ_1 from (3). For the nonnegative case such a modification consists of adding the positivity constraints on the elements of the unknown \mathbf{x} (we typically call such a modification of the standard ℓ_1 , the nonnegative ℓ_1). In a similar fashion, for the binary sparse case the following modification of (3) is typically considered (see e.g. [7, 25])

$$\begin{aligned} \min \quad & \|\mathbf{x}\|_1 \\ \text{subject to} \quad & A\mathbf{x} = \mathbf{y} \\ & 0 \leq \mathbf{x}_i \leq 1, 1 \leq i \leq n. \end{aligned} \tag{4}$$

The above problem, to which we will refer as the *binary* or *box* ℓ_1 , is fairly similar to the standard ℓ_1 from (3). When it comes to the binary vectors (similarly to what was the case for the nonnegative vectors), one expects that (4) should have a bit better recovery abilities than the standard ℓ_1 as it incorporates the a priori available knowledge that the elements of the unknown sparse vectors are constrained to be in $[0, 1]$ interval (in fact, not only should it have better recovery abilities than the standard ℓ_1 , it should actually have better recovery abilities than the nonnegative ℓ_1 as well). [25] rigorously showed that this is indeed

true. More importantly, in a statistical context, [25] precisely quantified by how much the algorithm from (4) improves on both, the standard and the nonnegative ℓ_1 . In the following sections we will in detail recall on the results from [25]. Here we briefly emphasize the difference between what was done in [25] and what will be done here. The results of [25] relate to the so-called phase-transition (PT) phenomena (these are of course the same phenomena that appeared in [4, 5, 22, 23] among the key properties that the standard and the nonnegative ℓ_1 exhibit). Basically, in the standard linear regime (regime where n is large, $m = \alpha n$, $k = \beta n$, and α and β are constants independent of n) [25] precisely characterized the so-called “breaking points” where these phase transitions happen (essentially the highest possible β for which the solution of (4) with overwhelming probability matches the sparsest solution of (2) for a fixed α ; under overwhelming probability we will in this paper consider probability over statistics of A that is no more than a number exponentially decaying in n away from 1). On the other hand, here, we will rely on the concepts introduced in [17] and will take a look at the phase transitions from a different angle. Following [17], we will connect the phase transitions to the so-called *large deviations principle* (LDP) from the classical probability theory and provide their explicit characterizations when viewed in such a way. We will do so for the binary/box ℓ_1 from (4) when used as a heuristic for finding two types of sparse unknown vectors constrained to have elements from a real interval: the first one being the above introduced binary sparse vectors and the second one being the so-called box-constrained vectors that we will introduce later on. Moreover, we will do so through two seemingly different approaches, one that is purely probabilistic and another one that has a nice connection to the high-dimensional geometry.

We will split the presentation into several sections, but two of them will of course be dominant. We will start by discussing the phase transitions of the binary ℓ_1 . After that we will move to the LDP characterizations and their connections with the PTs. In the later sections of the paper we will show how the PT and LDP results that we will create for the ℓ_1 from (4) when used for finding the binary sparse vectors can be modified so that they fit the usage of such ℓ_1 for finding the above mentioned box-constrained sparse vectors.

2 Binary ℓ_1

In this section we will revisit the phase transitions (PTs) of the ℓ_1 from (4) and then we will in great detail study the corresponding LDPs. From this point on we will make a clear distinction in the used terminology when it comes to the binary and box ℓ_1 . Namely, we will exclusively refer to the ℓ_1 from (4), the *binary* ℓ_1 , when it is used for solving systems known to have binary solutions. On the other hand, the term *box* ℓ_1 will be exclusively reserved for the usage of the ℓ_1 from (4) for solving systems known to have box-constrained solutions which, as mentioned earlier, we will introduce later on.

2.1 Phase transitions

Naturally, we start by recalling on the definitions of the PTs. These are of course generally well known, so we briefly state them without too much detailing (for a more comprehensive view, a long line of our work [14, 17, 22, 23, 25] can be consulted). To that end, we say that for any given constant $0 < \alpha \leq 1$ and *any* given binary \mathbf{x} with a given fixed location of its nonzero components there will be a maximum allowable value of β such that (4) finds that given \mathbf{x} with overwhelming probability. We will refer to this maximum allowable value of β as the *weak threshold/breaking point* and will denote it by β_w (see, e.g. [17, 20, 23, 24, 26]). Correspondingly, we also say that the algorithm exhibits the *weak* phase transition (i.e. *weak* PT). Under fully characterizing the weak phase transition one then considers determining the so-called *weak* PT curve in (α, β) plane so that for any pair (α, β) that is below this curve the algorithm (here (4)) succeeds with overwhelming probability in solving (2); otherwise it fails. In addition to the weak phase transitions, one can define various other forms of phase transitions. However, we stop short of discussing these in greater details as they will not be the main subject of this paper (more on them though can be found in e.g. [5, 16, 17, 21–23]).

As mentioned earlier and as is by now well known, [4, 5, 22, 23] fully characterized the standard ℓ_1 PT ([4, 5] through a high-dimensional geometry and [22, 23] through a purely probabilistic approach). In [25] we went a step further and fully characterized the binary ℓ_1 PT as well. The following theorem summarizes the obtained characterization.

Theorem 1. ([25] *Exact binary ℓ_1 ’s weak threshold/PT*) *Let A be an $m \times n$ matrix in (2) with i.i.d.*

standard normal components. Let the unknown \mathbf{x} that solves (2) be binary k -sparse. Further, let the locations of nonzero elements of \mathbf{x} be arbitrarily chosen but fixed. Assume that the nonzero elements of \mathbf{x} are equal to one. Let k, m, n be large and let $\alpha_w = \frac{m}{n}$ and $\beta_w = \frac{k}{n}$ be constants independent of m and n . Let erfinv be the inverse of the standard error function associated with zero-mean unit variance Gaussian random variable. Further, let α_w and β_w satisfy the following **fundamental characterization of the binary ℓ_1 's PT**

$$\xi_{\alpha_w}^{(bin)}(\beta_w) \triangleq \psi_{\beta_w}^{(bin)}(\alpha_w) \triangleq \frac{(1-2\beta_w)\sqrt{\frac{2}{\pi}}e^{-\left(\text{erfinv}\left(\frac{1-2\alpha_w}{1-2\beta_w}\right)\right)^2}}{2\alpha_w\sqrt{2}\text{erfinv}\left(\frac{1-2\alpha_w}{1-2\beta_w}\right)} = 1. \quad (5)$$

Then:

1. If $\alpha > \alpha_w$ then with overwhelming probability the solution of (4) is the binary k -sparse \mathbf{x} that solves (2).
2. If $\alpha < \alpha_w$ then with overwhelming probability the binary k -sparse \mathbf{x} with given fixed locations of nonzero components is the solution of (2) and is **not** the solution of (4).

Proof. The first part was established in [25] and the second one followed automatically from considerations in [19, 22]. \square

2.1.1 Doubling the number of equations

The fundamental PT characterizations given in the above theorem are indeed well defined. Namely, for any given fixed $\alpha \in (0, \frac{1}{2})$ there will be a unique $\beta \in (0, \alpha)$ such that $\xi_{\alpha}^{(bin)}(\beta) = 1$ and for any given fixed $\beta \in (0, \frac{1}{2})$ there will be a unique $\alpha \in (\beta, \frac{1}{2})$ such that $\psi_{\beta}^{(bin)}(\alpha) = 1$. This follows immediately after one first notes that the change $\beta \leftarrow 2\beta$ and $\alpha \leftarrow 2\alpha$ transforms the above characterizations into the corresponding ones obtained for the standard ℓ_1 in [17, 22, 23] and then recalls that in [17] it was explicitly shown that these characterizations are indeed unambiguous. What is perhaps a bit more interesting (especially from a practical point of view) is the so-called doubling the number of equations phenomenon. Namely, as the above mentioned change $\beta \leftarrow 2\beta$, $\alpha \leftarrow 2\alpha$ indicates, the binary ℓ_1 for the same $\frac{\beta}{\alpha}$ ratio needs exactly two times smaller number of equations. This can be clearly seen from Figure 1 where we show the theoretical PT curves for the binary ℓ_1 that one can obtain based on (5). In addition to the binary ℓ_1 PT curve we also show the corresponding PT curves for the standard and nonnegative ℓ_1 PTs. As arrows in Figure 1 indicate, to achieve the same β/α ratio that the binary ℓ_1 achieves, the standard ℓ_1 needs exactly two times larger α .

2.2 Large deviations

In this section we discuss the binary ℓ_1 LDP characterizations that will provide a significantly richer spectrum of information about the above discussed PTs – namely, they will explain the algorithms behaviour not only at the breaking points/thresholds but also in the entire transition zone around these points. The key difference between the standard PTs and the LDPs that we will discuss below will be in the exactness of the characterizations of the rates at which the probabilities of algorithms' success (failure) tend to zero as the systems dimensions deviate from the ones that satisfy the PT curves (i.e. the breaking points of the algorithms' success). To achieve full exactness in characterizing these rates, we will below determine the LDPs relying on the connection between the PTs and the LDPs that we established in [17]. Consequently, we will also try to emulate the strategies designed in [17]. Moreover, to make the exposition easier to follow, we will try to make everything look as parallel to what was done in [17] as possible (many repetitive steps though will be skipped and the emphasis will be on those that bring the key differences).

As is usual the case with many of the strategies that we designed, we start things off by recalling on a couple of fundamentally important technical results that we established in [23–26]. To ensure the clarity and simplicity of the exposition, we will without loss of generality assume that the elements $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{n-k}$ of \mathbf{x} are equal to zero and that the elements $\mathbf{x}_{n-k+1}, \mathbf{x}_{n-k+2}, \dots, \mathbf{x}_n$ are all equal to one (we emphasize that it is of course not known to the algorithm beforehand which elements are equal to one, however it is assumed to be known that each element of the unknown vector \mathbf{x} in (4) is either zero or one; the above assumption is

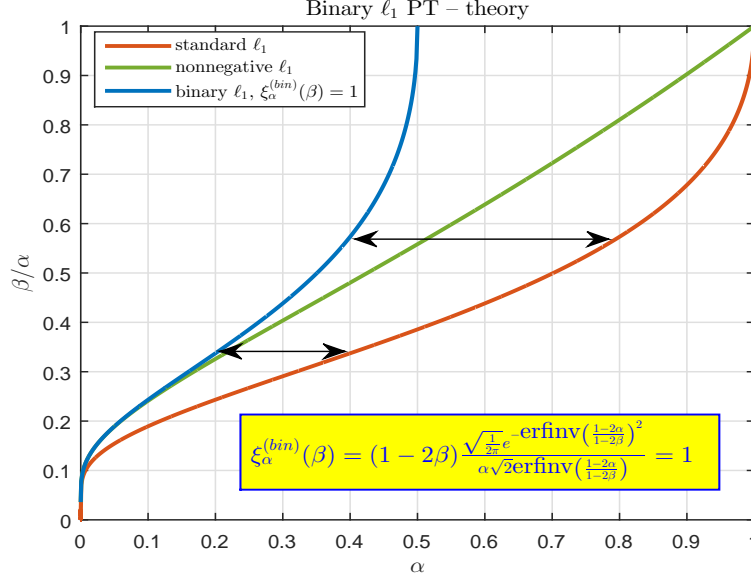


Figure 1: Binary ℓ_1 's weak PT; $\{(\alpha, \beta) | \xi_\alpha^{(bin)}(\beta) = 1\}$

of course only for the concreteness purposes of the analysis that will be presented below and is of course in an agreement with the requirement that the weak phase transition imposes). Relying on the breakthrough observations of [23, 24], we in [25] established the following theorem which is one of the key engines behind the entire machinery developed in [23–25].

Theorem 2. ([23–25] Nonzero elements of binary \mathbf{x} have fixed location) Assume that an $m \times n$ system matrix A is given. Let \mathbf{x} be a binary k sparse vector. Also let $\mathbf{x}_1 = \mathbf{x}_2 = \dots = \mathbf{x}_{n-k} = 0$. Assume that the nonzero elements of \mathbf{x} are equal to one. Further, assume that $\mathbf{y} \triangleq A\mathbf{x}$ and that \mathbf{w} is an $n \times 1$ vector such that $\mathbf{w}_i \geq 0, 1 \leq i \leq n-k$, and $\mathbf{w}_i \leq 0, n-k+1 \leq i \leq n$. If

$$(\forall \mathbf{w} \in \mathbf{R}^n | A\mathbf{w} = 0) \quad - \quad \sum_{i=n-k+1}^n \mathbf{w}_i < \sum_{i=1}^{n-k} \mathbf{w}_i, \quad (6)$$

then the solutions of (2) and (4) coincide. Moreover, if

$$(\exists \mathbf{w} \in \mathbf{R}^n | A\mathbf{w} = 0) \quad - \quad \sum_{i=n-k+1}^n \mathbf{w}_i \geq \sum_{i=1}^{n-k} \mathbf{w}_i, \quad (7)$$

then the solution of (2) and is not the solution of (4).

To facilitate the exposition we set

$$S_w \triangleq \{\mathbf{w} \in S^{n-1} | \quad - \quad \sum_{i=n-k+1}^n \mathbf{w}_i < \sum_{i=1}^{n-k} \mathbf{w}_i, \quad \mathbf{w}_i \geq 0, 1 \leq i \leq (n-k), \quad \text{and} \quad \mathbf{w}_i \leq 0, (n-k)+1 \leq i \leq n-k\}, \quad (8)$$

and as in [17], we first provide a detailed analysis of the so-called upper tail of the LDP characterizations (as for the standard ℓ_1 LDPs, it will turn out that the minimal adaptations of the upper tail analysis automatically settle the lower tail as well).

2.2.1 Upper tail

We will first consider the LDPs upper tail, which means, the points (α, β) such that $\alpha \geq \alpha_w$ where α_w is such that $\psi_\beta^{(bin)}(\alpha_w) = \xi_{\alpha_w}^{(bin)}(\beta) = 1$. Assuming that the elements of A are i.i.d. standard normals and following [17], we write

$$P_{err} \triangleq P(\min_{\mathbf{w} \in S_w} \|\mathbf{A}\mathbf{w}\|_2 \leq 0) = P(\max_{\mathbf{w} \in S_w} \min_{\|\mathbf{y}\|_2=1} (\mathbf{y}^T \mathbf{A}\mathbf{w}) \geq 0) \leq \min_{c_3 \geq 0} e^{-\frac{c_3^2}{2}} E e^{-c_3 \|\mathbf{g}\|_2} E e^{c_3 w(\mathbf{h}, S_w)}, \quad (9)$$

where P_{err} is the so-called probability of error/failure, i.e. the probability that (4) fails to produce the solution of (2) and

$$\begin{aligned} w(\mathbf{h}, S_w) &\triangleq \max_{\mathbf{w} \in S_w} (\mathbf{h}^T \mathbf{w}) = \max_{\bar{\mathbf{y}} \in \mathbb{R}^n} \sum_{i=1}^n \mathbf{h}_i \bar{\mathbf{y}}_i \\ &\text{subject to} \quad \bar{\mathbf{y}}_i \geq 0, 0 \leq i \leq n \\ &\quad \sum_{i=n-k+1}^n \bar{\mathbf{y}}_i \geq \sum_{i=1}^{n-k} \bar{\mathbf{y}}_i \\ &\quad \sum_{i=1}^n \bar{\mathbf{y}}_i^2 \leq 1, \end{aligned} \quad (10)$$

with the elements of \mathbf{h} being the i.i.d. standard normals. As in [14, 16–18, 20, 23, 25, 26] one writes

$$\begin{aligned} w(\mathbf{h}, S_w) &= - \max_{\nu \geq 0, \gamma \geq 0} \min_{\bar{\mathbf{y}}} \sum_{i=1}^n -\mathbf{h}_i \bar{\mathbf{y}}_i + \nu \sum_{i=1}^{n-k} \bar{\mathbf{y}}_i - \nu \sum_{i=n-k+1}^n \bar{\mathbf{y}}_i + \gamma \sum_{i=1}^n \bar{\mathbf{y}}_i^2 - \gamma \\ &\text{subject to} \quad \bar{\mathbf{y}}_i \geq 0, 0 \leq i \leq n, \end{aligned} \quad (11)$$

and finally

$$\begin{aligned} w(\mathbf{h}, S_w) &= \min_{\nu \geq 0, \gamma \geq 0} \frac{\sum_{i=1}^{n-k} \max(\mathbf{h}_i - \nu, 0)^2 + \sum_{i=n-k+1}^n \max(\mathbf{h}_i + \nu, 0)^2}{4\gamma} + \gamma \\ &= \min_{\nu \geq 0} \sqrt{\sum_{i=1}^{n-k} \max(\mathbf{h}_i - \nu, 0)^2 + \sum_{i=n-k+1}^n \max(\mathbf{h}_i + \nu, 0)^2}. \end{aligned} \quad (12)$$

We summarize the above methodology to upper bound P_{err} in the following theorem.

Theorem 3. *Let A be an $m \times n$ matrix in (2) with i.i.d. standard normal components. Let the unknown \mathbf{x} in (2) be binary k -sparse and let the locations of nonzero elements of \mathbf{x} be arbitrarily chosen but fixed. Assume that the nonzero elements of \mathbf{x} are equal to one. Let P_{err} be the probability that the solution of (4) is not the solution of (2). Then*

$$P_{err} \leq \min_{c_3 \geq 0} e^{-\frac{c_3^2}{2}} e^{-c_3 \|\mathbf{g}\|_2} E e^{c_3 w(\mathbf{h}, S_w)} = \min_{c_3 \geq 0} \left(e^{-\frac{c_3^2}{2}} \frac{1}{\sqrt{2\pi}^m} \int_{\mathbf{g}} e^{-\sum_{i=1}^m \mathbf{g}_i^2 / 2 - c_3 \|\mathbf{g}\|_2} d\mathbf{g} \min_{\nu \geq 0, \gamma \geq \frac{c_3}{2}} w_1^{n-k} w_2^k e^{c_3 \gamma} \right), \quad (13)$$

where

$$\begin{aligned} w_1 &= \frac{1}{\sqrt{2\pi}} \int_{\mathbf{h}} e^{-h^2/2} e^{c_3 \max(h-\nu, 0)^2/4/\gamma} dh = \frac{1}{2} \left(\frac{e^{\frac{c_3 \nu^2/4/\gamma}{1-c_3/2/\gamma}}}{\sqrt{1-c_3/2/\gamma}} \operatorname{erfc} \left(\frac{\nu}{\sqrt{2}\sqrt{1-c_3/2/\gamma}} \right) + \operatorname{erf} \left(\frac{\nu}{\sqrt{2}} \right) + 1 \right) \\ w_2 &= \frac{1}{\sqrt{2\pi}} \int_{\mathbf{h}} e^{-h^2/2} e^{c_3 \max(h+\nu, 0)^2/4/\gamma} dh = \frac{1}{2} \left(\frac{e^{\frac{c_3 \nu^2/4/\gamma}{1-c_3/2/\gamma}}}{\sqrt{1-c_3/2/\gamma}} \operatorname{erfc} \left(\frac{-\nu}{\sqrt{2}\sqrt{1-c_3/2/\gamma}} \right) + \operatorname{erf} \left(\frac{-\nu}{\sqrt{2}} \right) + 1 \right). \end{aligned}$$

(14)

Proof. Follows from the above considerations and ultimately through the mechanisms developed in [14, 16–18, 20, 23, 25, 26]. \square

The above theorem clearly provides an upper bound that holds for any integers m , k , and n (provided $k \leq m \leq n$ so that the results make sense). Below we will be interested in the LDP type of results which naturally assume the $n \rightarrow \infty$ asymptotic regime (the same is of course true for the PT types of results and Theorem 1). Following [17], we consider the decay rate of P_{err} , namely $I_{err}^{(bin)}(\alpha, \beta)$,

$$I_{err}^{(bin)}(\alpha, \beta) \triangleq \lim_{n \rightarrow \infty} \frac{\log P_{err}}{n}. \quad (15)$$

and based on Theorem 3 we have the following LDP type of theorem.

Theorem 4. Assume the setup of Theorem 3. Further, let integers m , k , and n be large ($k \leq m \leq n$) such that $\beta = \frac{k}{n}$ and $\alpha = \frac{m}{n}$ are constants independent of n . Assume that a pair (α, β) is given. Also, assume the following scaling: $c_3 \rightarrow c_3\sqrt{n}$ and $\gamma \rightarrow \gamma\sqrt{n}$. Then

$$\begin{aligned} I_{err}^{(bin)}(\alpha, \beta) &\triangleq \lim_{n \rightarrow \infty} \frac{\log P_{err}}{n} \\ &\leq \min_{c_3 \geq 0} \left(-\frac{(c_3)^2}{2} + I_{sph} + \min_{\nu \geq 0, \gamma \geq \frac{c_3}{2}} ((1 - \beta) \log w_1 + \beta \log w_2 + c_3 \gamma) \right) \triangleq I_{err,u}^{(bin,ub)}(\alpha, \beta), \end{aligned} \quad (16)$$

where

$$\begin{aligned} I_{sph} &= \hat{\gamma} c_3 - \frac{\alpha}{2} \log \left(1 - \frac{c_3}{2\hat{\gamma}} \right) \\ \hat{\gamma} &= \frac{c_3 - \sqrt{(c_3)^2 + 4\alpha}}{4} \\ w_1 &= \frac{1}{\sqrt{2\pi}} \int_h e^{-h^2/2} e^{c_3 \max(h-\nu, 0)^2/4/\gamma} dh = \frac{1}{2} \left(\frac{e^{\frac{c_3 \nu^2/4/\gamma}{1-c_3/2/\gamma}}}{\sqrt{1-c_3/2/\gamma}} \operatorname{erfc} \left(\frac{\nu}{\sqrt{2}\sqrt{1-c_3/2/\gamma}} \right) + \operatorname{erf} \left(\frac{\nu}{\sqrt{2}} \right) + 1 \right) \\ w_2 &= \frac{1}{\sqrt{2\pi}} \int_h e^{-h^2/2} e^{c_3 \max(h+\nu, 0)^2/4/\gamma} dh = \frac{1}{2} \left(\frac{e^{\frac{c_3 \nu^2/4/\gamma}{1-c_3/2/\gamma}}}{\sqrt{1-c_3/2/\gamma}} \operatorname{erfc} \left(\frac{-\nu}{\sqrt{2}\sqrt{1-c_3/2/\gamma}} \right) + \operatorname{erf} \left(\frac{-\nu}{\sqrt{2}} \right) + 1 \right). \end{aligned} \quad (17)$$

Proof. Follows in a fashion analogous to the one employed in [17]. \square

The above optimization problem can be solved numerically and that would be enough to provide the estimates for the rate of P_{err} 's decay. Instead of relying on a numerical solving we will below present an explicit solution. We will try to follow at least to some degree the methodology introduced in [17]. However, the technical considerations will be a bit more involved and our presentation will occasionally deviate from what was presented in [17]. Moreover, it will turn out that the optimizing quantities and eventually the LDPs rate functions will exhibit a behavior substantially different from the one observed in [17] when the standard ℓ_1 was considered.

2.2.2 Determining $I_{err,u}^{(bin,ub)}$

As in [17] we start by setting

$$A_0 \triangleq \sqrt{1 - \frac{c_3}{2\gamma}}, \quad (18)$$

and then consider the following optimization problem

$$I_{err,u}^{(bin,ub)}(\alpha, \beta) \triangleq \min_{c_3 \geq 0, \nu \geq 0, A_0 \leq 1} \zeta_{\alpha,\beta}(c_3, \nu, A_0), \quad (19)$$

where

$$\begin{aligned} \zeta_{\alpha,\beta}(c_3, \nu, A_0) &= \left(-\frac{c_3^2}{2} + I_{sph} + (1 - \beta) \log w_1 + \beta \log w_2 + \frac{c_3^2}{2(1 - A_0^2)} \right) \\ I_{sph} &= \hat{\gamma} c_3 - \frac{\alpha}{2} \log \left(1 - \frac{c_3}{2\hat{\gamma}} \right) \\ \hat{\gamma} &= \frac{c_3 - \sqrt{(c_3)^2 + 4\alpha}}{4} \\ w_1 &= \frac{1}{2} \left(\frac{e^{\frac{(1-A_0^2)\nu^2}{2A_0^2}}}{A_0} \operatorname{erfc} \left(\frac{\nu}{\sqrt{2}A_0} \right) + \operatorname{erf} \left(\frac{\nu}{\sqrt{2}} \right) + 1 \right) \\ w_2 &= \frac{1}{2} \left(\frac{e^{\frac{(1-A_0^2)\nu^2}{2A_0^2}}}{A_0} \operatorname{erfc} \left(\frac{-\nu}{\sqrt{2}A_0} \right) + \operatorname{erf} \left(\frac{-\nu}{\sqrt{2}} \right) + 1 \right). \end{aligned} \quad (20)$$

To solve the optimization problem in (19) we will compute the derivatives of $\zeta_{\alpha,\beta}(c_3, \nu, A_0)$ with respect to c_3 , ν , and A_0 and solve the following system of three equations

$$\frac{d\zeta_{\alpha,\beta}(c_3, \nu, A_0)}{dc_3} = \frac{d\zeta_{\alpha,\beta}(c_3, \nu, A_0)}{d\nu} = \frac{d\zeta_{\alpha,\beta}(c_3, \nu, A_0)}{dA_0} = 0. \quad (21)$$

We will in optimizations below consider what will refer to as the hard regime, i.e. we will consider (α, β) that ensure that the optima of the underlying optimizations are achieved nontrivially i.e. not on the boundaries (analysis of possible boundary optima is a small subcase and highly trivial compared to what we will present throughout the paper and we leave it as an easy exercise). Now, clearly, the above problem is fairly hard and at first glance it does not seem that there is much of a reason to believe that even after presumably lengthy computation of the above derivatives one would arrive anywhere close to the explicit solution. However, by a pure magic of mathematics this will turn out to be false and one can actually indeed solve the above system of equations. Before reaching the level where this will be clear a decent amount of patience may be needed. A few quick observations will also turn out to be very useful. We start with one of them that relates to the derivative with respect to c_3 and observe that for this derivative from [17] one immediately has

$$\frac{d\zeta_{\alpha,\beta}(c_3, \nu, A_0)}{dc_3} = -c_3 + \frac{c_3}{1 - A_0^2} + \frac{c_3 - \sqrt{(c_3)^2 + 4\alpha}}{2}. \quad (22)$$

Setting further the above derivative to zero implies

$$c_3 = \frac{(1 - A_0^2)\sqrt{\alpha}}{A_0}. \quad (23)$$

The derivatives with respect to ν and A_0 are a bit more involved. For the derivative with respect to ν we have

$$\frac{d\zeta_{\alpha,\beta}(c_3, \nu, A_0)}{d\nu} = \frac{d}{d\nu} \left(-\frac{c_3^2}{2} + I_{sph} + (1 - \beta) \log w_1 + \beta \log w_2 + \frac{c_3^2}{2(1 - A_0^2)} \right)$$

$$\begin{aligned}
&= \frac{1-\beta}{w_1} \left(\frac{(1-A_0^2)\nu}{A_0^2} e^{\frac{(1-A_0^2)\nu^2}{2A_0^2}} \operatorname{erfc}\left(\frac{\nu}{\sqrt{2}A_0}\right) - \frac{1-A_0^2}{A_0^2} \frac{2e^{-\frac{\nu^2}{2}}}{\sqrt{2}\sqrt{\pi}} \right) \\
&\quad - \frac{\beta}{w_2} \left(-\frac{(1-A_0^2)\nu}{A_0^2} e^{\frac{(1-A_0^2)\nu^2}{2A_0^2}} \operatorname{erfc}\left(\frac{-\nu}{\sqrt{2}A_0}\right) - \frac{1-A_0^2}{A_0^2} \frac{2e^{-\frac{\nu^2}{2}}}{\sqrt{2}\sqrt{\pi}} \right). \tag{24}
\end{aligned}$$

To facilitate the exposition we set

$$\begin{aligned}
y_1 &= \frac{\nu}{\sqrt{2}} \\
y_2 &= \frac{\nu}{\sqrt{2}A_0} = \frac{y_1}{A_0}. \tag{25}
\end{aligned}$$

Then we also set

$$\begin{aligned}
z_{1,\nu} &= \frac{(1-\beta)y_1 \left(\sqrt{2}y_2 e^{y_2^2} \operatorname{erfc}(y_2) - \sqrt{\frac{2}{\pi}} \right)}{y_2 e^{y_2^2} \operatorname{erfc}(y_2) + y_1 e^{y_1^2} \operatorname{erfc}(-y_1)} \\
z_{2,\nu} &= -\frac{\beta y_1 \left(-\sqrt{2}y_2 e^{y_2^2} \operatorname{erfc}(-y_2) - \sqrt{\frac{2}{\pi}} \right)}{y_2 e^{y_2^2} \operatorname{erfc}(-y_2) + y_1 e^{y_1^2} \operatorname{erfc}(y_1)}. \tag{26}
\end{aligned}$$

Combining (20), (24), (25), and (26) we obtain

$$\begin{aligned}
\frac{d\zeta_{\alpha,\beta}(c_3, \nu, A_0)}{d\nu} &= \frac{1-\beta}{w_1} \left(\frac{(1-A_0^2)\nu}{A_0^2} e^{\frac{(1-A_0^2)\nu^2}{2A_0^2}} \operatorname{erfc}\left(\frac{\nu}{\sqrt{2}A_0}\right) - \frac{1-A_0^2}{A_0^2} \frac{2e^{-\frac{\nu^2}{2}}}{\sqrt{2}\sqrt{\pi}} \right) \\
&\quad - \frac{\beta}{w_2} \left(-\frac{(1-A_0^2)\nu}{A_0^2} e^{\frac{(1-A_0^2)\nu^2}{2A_0^2}} \operatorname{erfc}\left(\frac{-\nu}{\sqrt{2}A_0}\right) - \frac{1-A_0^2}{A_0^2} \frac{2e^{-\frac{\nu^2}{2}}}{\sqrt{2}\sqrt{\pi}} \right) \\
&= \frac{1-A_0^2}{A_0^2} (z_{1,\nu} + z_{2,\nu}). \tag{27}
\end{aligned}$$

From (26) and (27) we then also have

$$\begin{aligned}
\frac{d\zeta_{\alpha,\beta}(c_3, \nu, A_0)}{d\nu} &= \frac{1-A_0^2}{A_0^2} (z_{1,\nu} + z_{2,\nu}) = 0 \\
\implies (z_{1,\nu} + z_{2,\nu}) &= \frac{(1-\beta)y_1 \left(\sqrt{2}y_2 e^{y_2^2} \operatorname{erfc}(y_2) - \sqrt{\frac{2}{\pi}} \right)}{y_2 e^{y_2^2} \operatorname{erfc}(y_2) + y_1 e^{y_1^2} \operatorname{erfc}(-y_1)} - \frac{\beta y_1 \left(-\sqrt{2}y_2 e^{y_2^2} \operatorname{erfc}(-y_2) - \sqrt{\frac{2}{\pi}} \right)}{y_2 e^{y_2^2} \operatorname{erfc}(-y_2) + y_1 e^{y_1^2} \operatorname{erfc}(y_1)} = 0. \tag{28}
\end{aligned}$$

Now we switch to the derivative with respect to A_0

$$\begin{aligned}
\frac{d\zeta_{\alpha,\beta}(c_3, \nu, A_0)}{dA_0} &= \frac{d}{dA_0} \left(-\frac{c_3^2}{2} + I_{sph} + (1-\beta) \log w_1 + \beta \log w_2 + \frac{c_3^2}{2(1-A_0^2)} \right) \\
&= (1-\beta) \frac{d \log w_1}{dA_0} + \beta \frac{d \log w_2}{dA_0} + \frac{c_3^2 A_0}{(1-A_0^2)^2} \\
&= (1-\beta) \frac{d \log w_1}{dA_0} + \beta \frac{d \log w_2}{dA_0} + \frac{\alpha}{A_0}. \tag{29}
\end{aligned}$$

From [17] we have

$$\frac{d \log w_1}{dA_0} = \frac{d \log \left(\frac{1}{A_0} e^{\frac{\nu^2}{2A_0^2}} \operatorname{erfc}\left(\frac{\nu}{\sqrt{2}A_0}\right) + e^{\frac{\nu^2}{2}} \left(\operatorname{erf}\left(\frac{\nu}{\sqrt{2}}\right) + 1 \right) \right)}{dA_0} = - \frac{e^{\frac{\nu^2}{2A_0^2}} (A_0^2 + \nu^2) \operatorname{erfc}\left(\frac{\nu}{\sqrt{2}A_0}\right) - \sqrt{\frac{2}{\pi}} A_0 \nu}{A_0^3 (e^{\frac{\nu^2}{2A_0^2}} \operatorname{erfc}\left(\frac{\nu}{\sqrt{2}A_0}\right) + A_0 e^{\frac{\nu^2}{2}} (\operatorname{erf}\left(\frac{\nu}{\sqrt{2}}\right) + 1))}. \quad (30)$$

Analogously to (30) we also have

$$\frac{d \log w_2}{dA_0} = \frac{d \log \left(\frac{1}{A_0} e^{\frac{\nu^2}{2A_0^2}} \operatorname{erfc}\left(\frac{-\nu}{\sqrt{2}A_0}\right) + e^{\frac{\nu^2}{2}} \left(\operatorname{erf}\left(\frac{-\nu}{\sqrt{2}}\right) + 1 \right) \right)}{dA_0} = - \frac{e^{\frac{\nu^2}{2A_0^2}} (A_0^2 + \nu^2) \operatorname{erfc}\left(\frac{-\nu}{\sqrt{2}A_0}\right) + \sqrt{\frac{2}{\pi}} A_0 \nu}{A_0^3 (e^{\frac{\nu^2}{2A_0^2}} \operatorname{erfc}\left(\frac{-\nu}{\sqrt{2}A_0}\right) + A_0 e^{\frac{\nu^2}{2}} (\operatorname{erf}\left(\frac{-\nu}{\sqrt{2}}\right) + 1))}. \quad (31)$$

Similarly to what was done in (26) we set

$$\begin{aligned} z_{1,A_0} &= -(1-\beta) \frac{y_2^2 \left((1+2y_2^2) e^{y_2^2} \operatorname{erfc}(y_2) - \sqrt{2} y_2 \sqrt{\frac{2}{\pi}} \right)}{y_1 (y_2 e^{y_2^2} \operatorname{erfc}(y_2) + y_1 e^{y_1^2} \operatorname{erfc}(-y_1))} \\ z_{2,A_0} &= -\beta \frac{y_2^2 \left((1+2y_2^2) e^{y_2^2} \operatorname{erfc}(-y_2) + \sqrt{2} y_2 \sqrt{\frac{2}{\pi}} \right)}{y_1 (y_2 e^{y_2^2} \operatorname{erfc}(-y_2) + y_1 e^{y_1^2} \operatorname{erfc}(y_1))}. \end{aligned} \quad (32)$$

Combining (24), (29), (30), (31), and (32) we obtain

$$\frac{d\zeta_{\alpha,\beta}(c_3, \nu, A_0)}{dA_0} = (1-\beta) \frac{d \log w_1}{dA_0} + \beta \frac{d \log w_2}{dA_0} + \frac{\alpha}{A_0} = z_{1,A_0} + z_{1,A_0} + \frac{\alpha y_2}{y_1}. \quad (33)$$

From (32) and (33) we also have

$$\begin{aligned} \frac{d\zeta_{\alpha,\beta}(c_3, \nu, A_0)}{dA_0} &= z_{1,A_0} + z_{1,A_0} + \frac{\alpha y_2}{y_1} = 0 \\ \Rightarrow & -(1-\beta) y_2 \frac{\left((1+2y_2^2) e^{y_2^2} \operatorname{erfc}(y_2) - \sqrt{2} y_2 \sqrt{\frac{2}{\pi}} \right)}{y_2 e^{y_2^2} \operatorname{erfc}(y_2) + y_1 e^{y_1^2} \operatorname{erfc}(-y_1)} - \beta y_2 \frac{\left((1+2y_2^2) e^{y_2^2} \operatorname{erfc}(-y_2) + \sqrt{2} y_2 \sqrt{\frac{2}{\pi}} \right)}{y_2 e^{y_2^2} \operatorname{erfc}(-y_2) + y_1 e^{y_1^2} \operatorname{erfc}(y_1)} + \alpha = 0. \end{aligned} \quad (34)$$

Combining further (28) and (34) we also have

$$\begin{aligned} \frac{d\zeta_{\alpha,\beta}(c_3, \nu, A_0)}{dA_0} &= z_{1,A_0} + z_{1,A_0} + \frac{\alpha y_2}{y_1} = 0 \\ \Rightarrow & -(1-\beta) y_2 \frac{e^{y_2^2} \operatorname{erfc}(y_2)}{y_2 e^{y_2^2} \operatorname{erfc}(y_2) + y_1 e^{y_1^2} \operatorname{erfc}(-y_1)} - \beta y_2 \frac{e^{y_2^2} \operatorname{erfc}(-y_2)}{y_2 e^{y_2^2} \operatorname{erfc}(-y_2) + y_1 e^{y_1^2} \operatorname{erfc}(y_1)} + \alpha = 0 \\ \Rightarrow & -(1-\beta) \frac{\sqrt{\frac{1}{\pi}}}{y_2 e^{y_2^2} \operatorname{erfc}(y_2) + y_1 e^{y_1^2} \operatorname{erfc}(-y_1)} + \beta \frac{\sqrt{\frac{1}{\pi}}}{y_2 e^{y_2^2} \operatorname{erfc}(-y_2) + y_1 e^{y_1^2} \operatorname{erfc}(y_1)} + \alpha = 0 \\ \Rightarrow & -\frac{1-\beta}{y_2 e^{y_2^2} \operatorname{erfc}(y_2) + y_1 e^{y_1^2} \operatorname{erfc}(-y_1)} + \frac{\beta}{y_2 e^{y_2^2} \operatorname{erfc}(-y_2) + y_1 e^{y_1^2} \operatorname{erfc}(y_1)} + \alpha \sqrt{\pi} = 0. \end{aligned} \quad (35)$$

From (28) we further have

$$\frac{\beta}{y_2 e^{y_2^2} \operatorname{erfc}(-y_2) + y_1 e^{y_1^2} \operatorname{erfc}(y_1)} = \frac{(1-\beta)}{(y_2 e^{y_2^2} \operatorname{erfc}(y_2) + y_1 e^{y_1^2} \operatorname{erfc}(-y_1))} \frac{\left(\sqrt{2} y_2 e^{y_2^2} \operatorname{erfc}(y_2) - \sqrt{\frac{2}{\pi}}\right)}{\left(-\sqrt{2} y_2 e^{y_2^2} \operatorname{erfc}(-y_2) - \sqrt{\frac{2}{\pi}}\right)}. \quad (36)$$

Combining further (35) and (36)

$$\begin{aligned} \frac{d\zeta_{\alpha,\beta}(c_3, \nu, A_0)}{dA_0} &= z_{1,A_0} + z_{1,A_0} + \frac{\alpha y_2}{y_1} = 0 \\ \Rightarrow -\frac{1-\beta}{y_2 e^{y_2^2} \operatorname{erfc}(y_2) + y_1 e^{y_1^2} \operatorname{erfc}(-y_1)} + \frac{\beta}{y_2 e^{y_2^2} \operatorname{erfc}(-y_2) + y_1 e^{y_1^2} \operatorname{erfc}(y_1)} + \alpha \sqrt{\pi} &= 0 \\ \Rightarrow \frac{1-\beta}{y_2 e^{y_2^2} \operatorname{erfc}(y_2) + y_1 e^{y_1^2} \operatorname{erfc}(-y_1)} \left(-1 + \frac{\left(\sqrt{2} y_2 e^{y_2^2} \operatorname{erfc}(y_2) - \sqrt{\frac{2}{\pi}}\right)}{\left(-\sqrt{2} y_2 e^{y_2^2} \operatorname{erfc}(-y_2) - \sqrt{\frac{2}{\pi}}\right)} \right) + \alpha \sqrt{\pi} &= 0 \\ \Rightarrow y_2 e^{y_2^2} \operatorname{erfc}(y_2) + y_1 e^{y_1^2} \operatorname{erfc}(-y_1) &= \frac{1-\beta}{\alpha} \left(\frac{y_2 e^{y_2^2} \operatorname{erfc}(-y_2) + y_2 e^{y_2^2} \operatorname{erfc}(y_2)}{\sqrt{\pi} y_2 e^{y_2^2} \operatorname{erfc}(-y_2) + 1} \right) \\ \Rightarrow y_1 e^{y_1^2} \operatorname{erfc}(-y_1) &= \frac{1-\beta}{\alpha} \left(\frac{2 y_2 e^{y_2^2}}{\sqrt{\pi} y_2 e^{y_2^2} \operatorname{erfc}(-y_2) + 1} \right) - y_2 e^{y_2^2} \operatorname{erfc}(y_2) \triangleq f_1(y_2; \alpha, \beta). \end{aligned} \quad (37)$$

One can also combine (35) and (36) in the following alternative way to obtain

$$\begin{aligned} \frac{d\zeta_{\alpha,\beta}(c_3, \nu, A_0)}{dA_0} &= z_{1,A_0} + z_{1,A_0} + \frac{\alpha y_2}{y_1} = 0 \\ \Rightarrow -\frac{1-\beta}{y_2 e^{y_2^2} \operatorname{erfc}(y_2) + y_1 e^{y_1^2} \operatorname{erfc}(-y_1)} + \frac{\beta}{y_2 e^{y_2^2} \operatorname{erfc}(-y_2) + y_1 e^{y_1^2} \operatorname{erfc}(y_1)} + \alpha \sqrt{\pi} &= 0 \\ \Rightarrow \frac{\beta}{y_2 e^{y_2^2} \operatorname{erfc}(-y_2) + y_1 e^{y_1^2} \operatorname{erfc}(y_1)} \left(1 - \frac{\left(-\sqrt{2} y_2 e^{y_2^2} \operatorname{erfc}(-y_2) - \sqrt{\frac{2}{\pi}}\right)}{\left(\sqrt{2} y_2 e^{y_2^2} \operatorname{erfc}(y_2) - \sqrt{\frac{2}{\pi}}\right)} \right) + \alpha \sqrt{\pi} &= 0 \\ \Rightarrow y_2 e^{y_2^2} \operatorname{erfc}(-y_2) + y_1 e^{y_1^2} \operatorname{erfc}(y_1) &= \frac{\beta}{\alpha} \left(\frac{y_2 e^{y_2^2} \operatorname{erfc}(-y_2) + y_2 e^{y_2^2} \operatorname{erfc}(y_2)}{-\sqrt{\pi} y_2 e^{y_2^2} \operatorname{erfc}(y_2) + 1} \right) \\ \Rightarrow y_1 e^{y_1^2} \operatorname{erfc}(y_1) &= \frac{\beta}{\alpha} \left(\frac{2 y_2 e^{y_2^2}}{-\sqrt{\pi} y_2 e^{y_2^2} \operatorname{erfc}(y_2) + 1} \right) - y_2 e^{y_2^2} \operatorname{erfc}(-y_2) = -f_1(-y_2; \alpha, 1-\beta). \end{aligned} \quad (38)$$

From (37) and (38) one also has

$$\frac{1 + \operatorname{erf}(y_1)}{1 - \operatorname{erf}(y_1)} = \frac{\operatorname{erfc}(-y_1)}{\operatorname{erfc}(y_1)} = -\frac{f_1(y_2; \alpha, \beta)}{f_1(-y_2; \alpha, 1-\beta)}. \quad (39)$$

Finally after solving over $\operatorname{erf}(y_1)$ we obtain

$$\begin{aligned} \operatorname{erf}(y_1) &= \frac{f_1(y_2; \alpha, \beta) + f_1(-y_2; \alpha, 1-\beta)}{f_1(y_2; \alpha, \beta) - f_1(-y_2; \alpha, 1-\beta)} \\ \Leftrightarrow y_1 &= \operatorname{erfinv} \left(\frac{f_1(y_2; \alpha, \beta) + f_1(-y_2; \alpha, 1-\beta)}{f_1(y_2; \alpha, \beta) - f_1(-y_2; \alpha, 1-\beta)} \right). \end{aligned} \quad (40)$$

A combination of (37) (or (38)) and (40) gives

$$2\text{erfinv}\left(\frac{f_1(y_2; \alpha, \beta) + f_1(-y_2; \alpha, 1 - \beta)}{f_1(y_2; \alpha, \beta) - f_1(-y_2; \alpha, 1 - \beta)}\right) \frac{e^{\text{erfinv}\left(\frac{f_1(y_2; \alpha, \beta) + f_1(-y_2; \alpha, 1 - \beta)}{f_1(y_2; \alpha, \beta) - f_1(-y_2; \alpha, 1 - \beta)}\right)^2}}{f_1(y_2; \alpha, \beta) - f_1(-y_2; \alpha, 1 - \beta)} = 1, \quad (41)$$

which can be used to determine y_2 . Once y_2 is determined one can obtain y_1 from (37) and (38) (basically (40)) and using (18), (23), and (25), ν , A_0 , c_3 , and γ from the following

$$\begin{aligned} \nu &= y_2 \sqrt{2} \\ A_0 &= \frac{y_1}{y_2} \\ c_3 &= \frac{(1 - A_0^2) \sqrt{\alpha}}{A_0} \\ \gamma &= \frac{c_3}{2(1 - A_0)}. \end{aligned} \quad (42)$$

For the above choice one can then finally compute $\zeta_{\alpha, \beta}(c_3, \nu, A_0)$ using (20) in the following way. First, we note that from [17] one has

$$I_{sph} = -\frac{(1 - A_0^2)\alpha}{2} + \alpha \log(A_0). \quad (43)$$

Now, using (37), (38), (42), and (43) we have

$$\begin{aligned} I_{err, u}^{(bun, ub)} = \zeta_{\alpha, \beta}(c_3, \nu, A_0) &= -\frac{c_3^2}{2} + I_{sph} + (1 - \beta) \log w_1 + \beta \log w_2 + \frac{c_3^2}{2(1 - A_0^2)} \\ &= -\frac{c_3^2}{2} - \frac{(1 - A_0^2)\alpha}{2} + \alpha \log(A_0) + (1 - \beta) \log w_1 + \beta \log w_2 + \frac{c_3^2}{2(1 - A_0^2)} \\ &= \alpha \log(A_0) + (1 - \beta) \log w_1 + \beta \log w_2 \\ &= \alpha \log\left(\frac{y_1}{y_2}\right) + (1 - \beta) \log\left(e^{y_2^2} y_2 \text{erfc}(y_2) + e^{y_1^2} y_1 \text{erfc}(-y_1)\right) \\ &\quad + \beta \log\left(e^{y_2^2} y_2 \text{erfc}(-y_2) + e^{y_1^2} y_1 \text{erfc}(y_1)\right) - y_1^2 - \log(y_1) - \log(2) \\ &= \alpha \log\left(\frac{y_1}{y_2}\right) + (1 - \beta) \log\left(\frac{1 - \beta}{\alpha (\sqrt{\pi} e^{y_2^2} y_2 \text{erfc}(-y_2) + 1)}\right) \\ &\quad + \beta \log\left(\frac{\beta}{\alpha (-\sqrt{\pi} e^{y_2^2} y_2 \text{erfc}(y_2) + 1)}\right) + y_2^2 - y_1^2 - \log\left(\frac{y_1}{y_2}\right). \\ &= (\alpha - 1) \log\left(\frac{y_1}{y_2}\right) + (1 - \beta) \log\left(\frac{1 - \beta}{\alpha (\sqrt{\pi} e^{y_2^2} y_2 \text{erfc}(-y_2) + 1)}\right) \\ &\quad + \beta \log\left(\frac{\beta}{\alpha (-\sqrt{\pi} e^{y_2^2} y_2 \text{erfc}(y_2) + 1)}\right) + y_2^2 - y_1^2. \end{aligned} \quad (44)$$

From the discussion above we have that the choice for ν , A_0 , and c_3 given in (41) and (42) ensures that (21) is satisfied. Examining further the properties of the underlying functions one can also argue that this choice is not only a stationary point, but also a global optimum in (19). We skip pursuing these considerations further here. Instead, we will connect what was obtained above to another set of optimizing quantities that we will obtain through a different set of considerations and present later on (in fact, quite a lot more will turn out to be true, not only will the choice for ν , A_0 , and c_3 given in (41) and (42) turn out to be precisely the one that solves the optimization in (19) but also precisely the one that determines $I_{err}^{(bin)}(\alpha, \beta)$). Here though, we would like to point out that in our view it is quite remarkable that given the hardness of the

initial optimization problem a closed form solution of (19) could still be obtained. We summarize the above results in the following theorem.

Theorem 5. Assume the setup of Theorem 4 and assume that a pair (α, β) is given. Let $\alpha > \alpha_w$ where α_w is such that $\psi_\beta^{(bin)}(\alpha_w) = \xi_{\alpha_w}^{(bin)}(\beta) = 1$. Set

$$f_1(y_2; \alpha, \beta) \triangleq \frac{1-\beta}{\alpha} \left(\frac{2y_2 e^{y_2^2}}{\sqrt{\pi} y_2 e^{y_2^2} \operatorname{erfc}(-y_2) + 1} \right) - y_2 e^{y_2^2} \operatorname{erfc}(y_2). \quad (45)$$

Also let y_2 and y_1 satisfy the following **fundamental characterizations of the binary ℓ_1 's LDP**:

$$2 \operatorname{erfinv} \left(\frac{f_1(y_2; \alpha, \beta) + f_1(-y_2; \alpha, 1-\beta)}{f_1(y_2; \alpha, \beta) - f_1(-y_2; \alpha, 1-\beta)} \right) \frac{e^{\operatorname{erfinv} \left(\frac{f_1(y_2; \alpha, \beta) + f_1(-y_2; \alpha, 1-\beta)}{f_1(y_2; \alpha, \beta) - f_1(-y_2; \alpha, 1-\beta)} \right)^2}}}{f_1(y_2; \alpha, \beta) - f_1(-y_2; \alpha, 1-\beta)} = 1. \quad (46)$$

and

$$y_1 = \operatorname{erfinv} \left(\frac{f_1(y_2; \alpha, \beta) + f_1(-y_2; \alpha, 1-\beta)}{f_1(y_2; \alpha, \beta) - f_1(-y_2; \alpha, 1-\beta)} \right). \quad (47)$$

Then

$$\begin{aligned} I_{err}^{(bin)}(\alpha, \beta) \triangleq \lim_{n \rightarrow \infty} \frac{\log P_{err}}{n} \leq & (\alpha - 1) \log \left(\frac{y_1}{y_2} \right) + (1 - \beta) \log \left(\frac{1 - \beta}{\alpha (\sqrt{\pi} e^{y_2^2} y_2 \operatorname{erfc}(-y_2) + 1)} \right) \\ & + \beta \log \left(\frac{\beta}{\alpha (-\sqrt{\pi} e^{y_2^2} y_2 \operatorname{erfc}(y_2) + 1)} \right) + y_2^2 - y_1^2 \triangleq I_{ldp}^{(bin)}(\alpha, \beta). \end{aligned} \quad (48)$$

Moreover, for the above choice of y_2 and y_1 , ν , A_0 , c_3 , and γ in (42) achieve an optimum in (19) (and ultimately in (16)).

Proof. Follows from the above discussion. \square

The above results for the upper tail of the binary ℓ_1 LDP, remain correct in the lower tail regime. For the completeness we in the following section provide a short argument to confirm that this is indeed correct.

2.2.3 Lower tail

In this section we will quickly formalize the above statements about the lower tail type of large deviations. We rely on the strategy introduced in [17] and quickly have

$$P_{cor} \leq \min_{t_1} \min_{c_3 \geq 0} E e^{c_3 \|\mathbf{g}\|^2} E e^{-c_3 w(\mathbf{h}, S_w)} e^{-c_3 t_1} / P(g \geq t_1), \quad (49)$$

where obviously $P_{cor} = 1 - P_{err}$ is the probability that (4) does produce the solution of (2). We then have for the rate of P_{cor} 's decay

$$I_{cor}^{(bin)}(\alpha, \beta) \triangleq \lim_{n \rightarrow \infty} \frac{\log P_{cor}}{n}. \quad (50)$$

The following theorem is then the lower tail analogue to Theorem 4.

Theorem 6. Assume the setup of Theorem 4. Then

$$I_{cor}^{(bin)}(\alpha, \beta) \triangleq \lim_{n \rightarrow \infty} \frac{\log P_{cor}}{n}$$

$$\leq \min_{c_3 \geq 0} \left(-\frac{c_3^2}{2} + I_{sph}^+ + \max_{\nu \geq 0, \gamma^{(s)} \geq 0} ((1 - \beta \log w_1 + \beta \log w_2 - c_3 \gamma)) \right) \triangleq I_{cor,l}^{(p,ub)}(\alpha, \beta), \quad (51)$$

where

$$\begin{aligned} I_{sph}^+ &= \widehat{\gamma}_+ c_3 - \frac{\alpha d}{2} \log \left(1 - \frac{c_3}{2\widehat{\gamma}_+} \right) \\ \widehat{\gamma}_+ &= \frac{2c_3 + \sqrt{4c_3^2 + 16\alpha}}{8} \\ w_1 &= \frac{1}{2} \left(\frac{1}{\sqrt{2\pi}} \int_h e^{-h^2/2} e^{-c_3 \max(h-\nu, 0)^2/4/\gamma} dh = \frac{e^{\frac{-c_3 \nu^2/4/\gamma}{1+c_3/2/\gamma}}}{\sqrt{1+c_3/2/\gamma}} \operatorname{erfc} \left(\frac{\nu}{\sqrt{2}\sqrt{1+c_3/2/\gamma}} \right) + \operatorname{erf} \left(\frac{\nu}{\sqrt{2}} \right) + 1 \right) \\ w_2 &= \frac{1}{2} \left(\frac{1}{\sqrt{2\pi}} \int_h e^{-h^2/2} e^{-c_3 \max(h+\nu, 0)^2/4/\gamma} dh = \frac{e^{\frac{-c_3 \nu^2/4/\gamma}{1+c_3/2/\gamma}}}{\sqrt{1+c_3/2/\gamma}} \operatorname{erfc} \left(\frac{-\nu}{\sqrt{2}\sqrt{1+c_3/2/\gamma}} \right) + \operatorname{erf} \left(\frac{-\nu}{\sqrt{2}} \right) + 1 \right). \end{aligned} \quad (52)$$

Proof. Follows in exactly the same way as the result for the lower tail of the standard ℓ_1 LDP in [17]. \square

Instead of repeating the procedure from the previous section to solve the above optimization problem, one can just quickly observe that the change $c_3 \rightarrow -c_3$ gives as in Section 2.2.2

$$A_0 \triangleq \sqrt{1 - \frac{c_3}{2\gamma}}, \quad (53)$$

and

$$I_{cor,l}^{(p,ub)}(\alpha, \beta) \triangleq \min_{c_3 \leq 0} \max_{\nu \geq 0, A_0 \leq 1} \zeta_{\alpha, \beta}(c_3, \nu, A_0) \quad (54)$$

where

$$\begin{aligned} \zeta_{\alpha, \beta}(c_3, \nu, A_0) &= \left(-\frac{c_3^2}{2} + I_{sph}^+ + (1 - \beta) \log w_1 + \beta \log w_2 + \frac{c_3^2}{2(1 - A_0^2)} \right) \\ I_{sph}^+ &= -\widehat{\gamma}^+ c_3 - \frac{\alpha}{2} \log \left(1 + \frac{c_3}{2\widehat{\gamma}^+} \right) \\ \widehat{\gamma}^+ &= \frac{-c_3 + \sqrt{c_3^2 + 4\alpha}}{4} = -\widehat{\gamma}, \end{aligned} \quad (55)$$

and w_1 and w_2 are as in (20). Also, $\zeta_{\alpha, \beta}(c_3, \nu, A_0)$ defined in (55) is exactly the same as the corresponding one in (20) which means that one can proceed with the computation of all the derivatives as earlier and the values we have chosen for c_3 , ν , γ , and A_0 in the upper tail regime will have the same form. The following theorem summarizes the final results (this is of course nothing but a lower tail analogue to Theorem 5).

Theorem 7. Assume the setup of Theorem 5 and assume that a pair (α, β) is given. Differently from Theorem 5, let $\alpha < \alpha_w$ where α_w is such that $\psi_\beta^{(bin)}(\alpha_w) = \xi_{\alpha_w}^{(bin)}(\beta) = 1$. Also let y_2 and y_1 satisfy the **fundamental binary ℓ_1 's LDP characterizations** as in Theorem 5. Then choosing ν , c_3 , and γ in the optimization problem in (51) as ν , $-c_3$, and γ from Theorem 5 (or equivalently, choosing ν , c_3 , and A_0 in the optimization problem in (54) as ν , c_3 , and A_0 from Theorem 5) gives needed $\zeta_{\alpha, \beta}(c_3, \nu, A_0)$.

Proof. Follows from the considerations leading to Theorem 5. \square

Clearly, in terms of the translation of the upper tail results to the lower tail, there is really not much difference compared to the standard ℓ_1 and its considerations from [17]. Along the same lines, here we will have for $\alpha < \alpha_w$, $y_1 > y_2$ which means $A_0 > 1$ and finally $c_3 < 0$. In the upper tail regime (i.e. in Theorem 5) the reasoning is reversed and $c_3 > 0$.

2.3 High-dimensional geometry

The previous section introduced a purely probabilistic approach to deal with the LDPs. In this section though, we take a different path and analyze the LDPs via a high-dimensional integral geometry approach. As in the previous section, we will here again mostly focus on the upper tail regime (the results for the lower tail will automatically follow). In mathematical terms, we again assume that we are given a pair (α, β) and that $\alpha > \alpha_w$ (where α_w is such that $\psi_\beta^{(bin)}(\alpha_w) = \xi_{\alpha_w}^{(bin)}(\beta) = 1$). We will rely on the following observations from [15]

$$\Psi_{net}^{(bin)}(\alpha, \beta) = I_{err}^{(bin)}(\alpha, \beta) \triangleq \lim_{n \rightarrow \infty} \frac{\log P_{err}}{n} = \max_{\gamma_g \in (0, \min(1-\alpha, \beta))} (\Psi_{com} + \Psi_{int} + \Psi_{ext}), \quad (56)$$

where

$$\begin{aligned} H(x) &= x \log(x) + (1-x) \log(1-x) \\ \Psi_{com} &= -(1-\beta)H\left(\frac{1-\alpha-\gamma_g}{1-\beta}\right) - \beta H\left(\frac{\beta-\gamma_g}{\beta}\right) \\ \Psi_{int} &= \min_{y_i \geq 0} (\alpha y_i^2 + (\alpha - \beta + \gamma_g) \log(\operatorname{erfc}(y_i)) + (\beta - \gamma_g) \log(\operatorname{erfc}(-y_i))) - \alpha \log(2) \\ \Psi_{ext} &= \max_{y_e \geq 0} (-\alpha y_e^2 + (1 - \alpha - \gamma_g) \log(\operatorname{erfc}(-y_e)) + \gamma_g \log(\operatorname{erfc}(y_e))) - (1 - \alpha) \log(2). \end{aligned} \quad (57)$$

One can of course solve the above problem numerically. Here, we will raise the bar a bit higher and look for an explicit characterization of the solution. To that end we start with the following transformation of the above optimization problem

$$\max_{\gamma_g \in (0, \min(1-\alpha, \beta)), y_e \geq 0} \min_{y_i \geq 0} \zeta_{\alpha, \beta}(\gamma_g, y_e, y_i) \quad (58)$$

where

$$\begin{aligned} \zeta_{\alpha, \beta}(\gamma_g, y_e, y_i) &= -(1-\beta)H\left(\frac{1-\alpha-\gamma_g}{1-\beta}\right) - \beta H\left(\frac{\beta-\gamma_g}{\beta}\right) \\ &\quad + (\alpha y_i^2 + (\alpha - \beta + \gamma_g) \log(\operatorname{erfc}(y_i)) + (\beta - \gamma_g) \log(\operatorname{erfc}(-y_i))) - \alpha \log(2) \\ &\quad + (-\alpha y_e^2 + (1 - \alpha - \gamma_g) \log(\operatorname{erfc}(-y_e)) + \gamma_g \log(\operatorname{erfc}(y_e))) - (1 - \alpha) \log(2). \end{aligned} \quad (59)$$

Before proceeding further, we will below establish a few nice properties of $\zeta_{\alpha, \beta}(\gamma_g, y_e, y_i)$. Namely, that for any fixed $(\alpha, \beta) \in (0, 1) \times (0, \alpha)$, we will show that $\zeta_{\alpha, \beta}(\gamma_g, y_e, y_i)$ is concave in γ_g and y_e and convex in y_i in the optimizing domain of interest.

2.3.1 Concavity in γ_g

We start by computing the first derivative with respect to γ_g

$$\begin{aligned} \frac{d\zeta_{\alpha, \beta}(\gamma_g, y_e, y_i)}{d\gamma_g} &= -(1-\beta) \frac{dH\left(\frac{1-\alpha-\gamma_g}{1-\beta}\right)}{d\gamma_g} - \beta \frac{dH\left(\frac{\beta-\gamma_g}{\beta}\right)}{d\gamma_g} + \log\left(\frac{\operatorname{erfc}(y_i)\operatorname{erfc}(y_e)}{\operatorname{erfc}(-y_i)\operatorname{erfc}(-y_e)}\right) \\ &= \log\left(\frac{\frac{1-\alpha-\gamma_g}{1-\beta}}{1 - \frac{1-\alpha-\gamma_g}{1-\beta}}\right) + \log\left(\frac{\frac{\beta-\gamma_g}{\beta}}{1 - \frac{\beta-\gamma_g}{\beta}}\right) + \log\left(\frac{\operatorname{erfc}(y_i)\operatorname{erfc}(y_e)}{\operatorname{erfc}(-y_i)\operatorname{erfc}(-y_e)}\right) \\ &= \log\left(\frac{1-\alpha-\gamma_g}{\alpha-\beta+\gamma_g}\right) + \log\left(\frac{\beta-\gamma_g}{\gamma_g}\right) + \log\left(\frac{\operatorname{erfc}(y_i)\operatorname{erfc}(y_e)}{\operatorname{erfc}(-y_i)\operatorname{erfc}(-y_e)}\right). \end{aligned} \quad (60)$$

We then also have for the second derivative

$$\begin{aligned}
\frac{d^2 \zeta_{\alpha, \beta}(\gamma_g, y_e, y_i)}{d\gamma_g^2} &= \frac{d \left(\log \left(\frac{1-\alpha-\gamma_g}{\alpha-\beta+\gamma_g} \right) + \log \left(\frac{\beta-\gamma_g}{\gamma_g} \right) + \log \left(\frac{\operatorname{erfc}(y_i) \operatorname{erfc}(y_e)}{\operatorname{erfc}(-y_i) \operatorname{erfc}(-y_e)} \right) \right)}{d\gamma_g} \\
&= \frac{-\frac{1-\alpha-\gamma_g}{(\alpha-\beta+\gamma_g)^2} - \frac{1}{\alpha-\beta+\gamma_g}}{\frac{1-\alpha-\gamma_g}{\alpha-\beta+\gamma_g}} + \frac{-\frac{\beta-\gamma_g}{\gamma_g^2} - \frac{1}{\gamma_g}}{\frac{\beta-\gamma_g}{\gamma_g}} \\
&= \frac{-(1-\beta)}{(1-\alpha-\gamma_g)(\alpha-\beta+\gamma_g)} + \frac{-\beta}{\gamma_g(\beta-\gamma_g)} \\
&< 0.
\end{aligned} \tag{61}$$

2.3.2 Convexity in y_i

To check convexity in y_i and concavity in y_e we will rely on the following

$$\frac{d \log(\operatorname{erfc}(x))}{dx} = -\frac{2e^{-x^2}}{\sqrt{\pi} \operatorname{erfc}(x)} \quad \text{and} \quad \frac{d \log(\operatorname{erfc}(-x))}{dx} = \frac{2e^{-x^2}}{\sqrt{\pi} \operatorname{erfc}(-x)}, \tag{62}$$

and

$$\frac{d^2 \log(\operatorname{erfc}(x))}{dx^2} = \frac{4e^{-x^2} \sqrt{\pi} x \operatorname{erfc}(x) - 4e^{-2x^2}}{\pi \operatorname{erfc}(x)^2} \quad \text{and} \quad \frac{d^2 \log(\operatorname{erfc}(-x))}{dx^2} = \frac{-4e^{-x^2} \sqrt{\pi} x \operatorname{erfc}(-x) - 4e^{-2x^2}}{\pi \operatorname{erfc}(-x)^2}. \tag{63}$$

As we will eventually need both, the first and the second derivative with respect to y_i , we find it convenient to compute them at the same time right here. For the first derivative we have

$$\begin{aligned}
\frac{d \zeta_{\alpha, \beta}(\gamma_g, y_e, y_i)}{dy_i} &= \frac{d(\alpha y_i^2 + (\alpha - \beta + \gamma_g) \log(\operatorname{erfc}(y_i)) + (\beta - \gamma_g) \log(\operatorname{erfc}(-y_i)))}{dy_i} \\
&= 2\alpha y_i - \frac{2(\alpha - \beta + \gamma_g) e^{-y_i^2}}{\sqrt{\pi} \operatorname{erfc}(y_i)} + \frac{2(\beta - \gamma_g) e^{-y_i^2}}{\sqrt{\pi} \operatorname{erfc}(-y_i)},
\end{aligned} \tag{64}$$

and for the second

$$\begin{aligned}
\frac{d^2 \zeta_{\alpha, \beta}(\gamma_g, y_e, y_i)}{dy_i^2} &= \frac{d^2(\alpha y_i^2 + (\alpha - \beta + \gamma_g) \log(\operatorname{erfc}(y_i)) + (\beta - \gamma_g) \log(\operatorname{erfc}(-y_i)))}{dy_i^2} \\
&= 2\alpha + (\alpha - \beta + \gamma_g) \frac{4e^{-y_i^2} \sqrt{\pi} y_i \operatorname{erfc}(y_i) - 4e^{-2y_i^2}}{\pi \operatorname{erfc}(y_i)^2} \\
&\quad + (\beta - \gamma_g) \frac{-4e^{-y_i^2} \sqrt{\pi} y_i \operatorname{erfc}(-y_i) - 4e^{-2y_i^2}}{\pi \operatorname{erfc}(-y_i)^2} \\
&= 2(\alpha - \beta + \gamma_g) \frac{\pi \operatorname{erfc}(y_i)^2 + 2e^{-y_i^2} \sqrt{\pi} y_i \operatorname{erfc}(y_i) - 2e^{-2y_i^2}}{\pi \operatorname{erfc}(y_i)^2} \\
&\quad + 2(\beta - \gamma_g) \frac{\pi \operatorname{erfc}(-y_i)^2 - 2e^{-y_i^2} \sqrt{\pi} y_i \operatorname{erfc}(-y_i) - 2e^{-2y_i^2}}{\pi \operatorname{erfc}(-y_i)^2}.
\end{aligned} \tag{65}$$

In [17] it was argued that $\left(\pi \operatorname{erfc}(y_i)^2 + 2e^{-y_i^2} \sqrt{\pi} y_i \operatorname{erfc}(y_i) - 2e^{-2y_i^2} \right) > 0$. In order to show that the above derivative is nonnegative it is then enough to show that $\left(\pi \operatorname{erfc}(-y_i)^2 - 2e^{-y_i^2} \sqrt{\pi} y_i \operatorname{erfc}(-y_i) - 2e^{-2y_i^2} \right)$ is an increasing function of y_i (this is technically needed only on $y_i \geq 0$). This will be automatically implied if $\left(\sqrt{\pi} \operatorname{erfc}(-y_i) - 2e^{-y_i^2} y_i \right)$ is increasing in y_i . The following simple argument shows that this is indeed the

case.

$$\frac{d\left(\sqrt{\pi}\operatorname{erfc}(-y_i) - 2e^{-y_i^2}y_i\right)}{dy_i} = 2e^{-y_i^2} - 2e^{-y_i^2} + 4y_i^2 \geq 0. \quad (66)$$

All of the above then implies that $\zeta_{\alpha,\beta}(\gamma_g, y_e, y_i)$ is indeed convex in y_i in the domain of interest (of course, a bit more is true based on the above but we stop short of discussing it as it goes beyond what is needed here).

2.3.3 Concavity in y_e

To check concavity in y_e we proceed as above. For the first derivative we have

$$\begin{aligned} \frac{d\zeta_{\alpha,\beta}(\gamma_g, y_e, y_i)}{dy_e} &= \frac{d(-\alpha y_e^2 + (1 - \alpha - \gamma_g)\log(\operatorname{erfc}(-y_e)) + \gamma_g \log(\operatorname{erfc}(y_e)))}{dy_e} \\ &= -2\alpha y_e + \frac{2(1 - \alpha - \gamma_g)e^{-y_e^2}}{\sqrt{\pi}\operatorname{erfc}(-y_e)} - \frac{2\gamma_g e^{-y_e^2}}{\sqrt{\pi}\operatorname{erfc}(y_e)}, \end{aligned} \quad (67)$$

and for the second

$$\begin{aligned} \frac{d^2\zeta_{\alpha,\beta}(\gamma_g, y_e, y_i)}{dy_e^2} &= \frac{d^2(-\alpha y_e^2 + (1 - \alpha - \gamma_g)\log(\operatorname{erfc}(-y_e)) + \gamma_g \log(\operatorname{erfc}(y_e)))}{dy_e^2} \\ &= -2\alpha + (1 - \alpha - \gamma_g) \frac{-4e^{-y_e^2}\sqrt{\pi}y_e\operatorname{erfc}(-y_e) - 4e^{-2y_e^2}}{\pi\operatorname{erfc}(-y_e)^2} \\ &\quad + \gamma_g \frac{4e^{-y_e^2}\sqrt{\pi}y_e\operatorname{erfc}(y_e) - 4e^{-2y_e^2}}{\pi\operatorname{erfc}(y_e)^2} \end{aligned} \quad (68)$$

If one can show that what multiplies γ_g in the above expression is not positive then the second derivative would also not be positive. To that end we have

$$-\gamma_g \frac{-e^{y_e^2}\sqrt{\pi}y_e\operatorname{erfc}(-y_e) - 1}{\pi\operatorname{erfc}(-y_e)^2} + \gamma_g \frac{e^{y_e^2}\sqrt{\pi}y_e\operatorname{erfc}(y_e) - 1}{\pi\operatorname{erfc}(y_e)^2} = \gamma_g \frac{(2\sqrt{\pi}y_e\operatorname{erfc}(-y_e)\operatorname{erfc}(y_e)e^{y_e^2} - 4\operatorname{erf}(y_e))}{\pi\operatorname{erfc}(y_e)^2\operatorname{erfc}(-y_e)^2}, \quad (69)$$

and if we show that $(2\sqrt{\pi}y_e\operatorname{erfc}(-y_e)\operatorname{erfc}(y_e)e^{y_e^2} - 4\operatorname{erf}(y_e)) \leq 0$ when $y_e \geq 0$ then the second derivative in (68) would not be positive. There are many ways how this can be shown. Here we rely on the following well known inequalities

$$\frac{2}{\sqrt{\pi}} \frac{e^{-y^2}}{y + \sqrt{y^2 + 2}} < \operatorname{erfc}(y) \leq \frac{2}{\sqrt{\pi}} \frac{e^{-y^2}}{y + \sqrt{y^2 + \frac{4}{\pi}}}. \quad (70)$$

Using (70) we have

$$2\sqrt{\pi}y_e\operatorname{erfc}(-y_e)\operatorname{erfc}(y_e)e^{y_e^2} - 4\operatorname{erf}(y_e) \leq \frac{4y_e\operatorname{erfc}(-y_e)}{y_e + \sqrt{y_e^2 + \frac{4}{\pi}}} - 4\operatorname{erf}(y_e) = \left(\frac{4\operatorname{erfc}(-y_e)}{1 + \sqrt{1 + \frac{4}{\pi y_e^2}}} - \frac{4\operatorname{erfc}(-y_e)}{\frac{1}{\operatorname{erf}(y_e)} + 1} \right). \quad (71)$$

From (71), we have that if $\text{erf}(y_e) \geq \frac{y_e}{\sqrt{y_e^2 + \frac{4}{\pi}}}$ then $(2\sqrt{\pi}y_e \text{erfc}(-y_e) \text{erfc}(y_e) e^{y_e^2} - 4\text{erf}(y_e)) \leq 0$. Based on (70) we have that

$$\frac{y_e}{\sqrt{y_e^2 + \frac{4}{\pi}}} \leq 1 - \frac{2}{\sqrt{\pi}} \frac{e^{-y_e^2}}{y_e + \sqrt{y_e^2 + \frac{4}{\pi}}} \quad (72)$$

implies

$$\text{erf}(y_e) \geq \frac{y_e}{\sqrt{y_e^2 + \frac{4}{\pi}}}. \quad (73)$$

Transforming (72) further we have

$$\begin{aligned} \frac{y_e}{\sqrt{y_e^2 + \frac{4}{\pi}}} &\leq 1 - \frac{2}{\sqrt{\pi}} \frac{e^{-y_e^2}}{y_e + \sqrt{y_e^2 + \frac{4}{\pi}}} \\ \iff e^{-y_e^2} &\leq \frac{2}{\sqrt{y_e^2 \pi + 4}} \\ \iff y_e^2 \frac{\pi}{4} + 1 &\leq e^{2y_e^2}. \end{aligned} \quad (74)$$

The last inequality holds and one then also has based on the above that (73) holds which based on (71) implies that the right side of the equality in (69) is not positive. This is then enough to conclude that the second derivative in (68) is not positive and $\zeta_{\alpha,\beta}(\gamma_g, y_e, y_i)$ is indeed concave in y_e on $y_e \geq 0$.

2.3.4 Solving the derivative equations

While the above properties of $\zeta_{\alpha,\beta}(\gamma_g, y_e, y_i)$ are nice and welcome one still needs to solve the following system of equations

$$\frac{d\zeta_{\alpha,\beta}(\gamma_g, y_e, y_i)}{d\gamma_g} = \frac{d\zeta_{\alpha,\beta}(\gamma_g, y_e, y_i)}{dy_e} = \frac{d\zeta_{\alpha,\beta}(\gamma_g, y_e, y_i)}{dy_i} = 0. \quad (75)$$

The derivatives with respect to γ_g , y_i , and y_e are computed in (60), (64), and (67), respectively. Setting the derivative in (64) to zero gives

$$\begin{aligned} \gamma_g \left(\frac{1}{\text{erfc}(y_i)} + \frac{1}{\text{erfc}(-y_i)} \right) &= \sqrt{\pi} \alpha e^{y_i^2} y_i - \frac{(\alpha - \beta)}{\text{erfc}(y_i)} + \frac{(\beta)}{\text{erfc}(-y_i)} \\ \iff \gamma_g &= \frac{\alpha}{2} \left(\sqrt{\pi} e^{y_i^2} y_i \text{erfc}(y_i) \text{erfc}(-y_i) - \text{erfc}(-y_i) \right) + \beta. \end{aligned} \quad (76)$$

After setting

$$A_{bin} = \frac{\alpha - \beta + \gamma_g}{1 - \alpha - \gamma_g} \frac{\gamma_g}{\beta - \gamma_g}, \quad (77)$$

from (60) one has

$$\begin{aligned} A_{bin} &= \frac{\text{erfc}(y_i) \text{erfc}(y_e)}{\text{erfc}(-y_i) \text{erfc}(-y_e)} \\ \iff \frac{\text{erfc}(-y_e)}{\text{erfc}(y_e)} &= \frac{\text{erfc}(y_i)}{\text{erfc}(-y_i) A_{bin}} \end{aligned}$$

$$\iff y_e = \operatorname{erfinv} \left(\frac{\operatorname{erfc}(y_i) - A_{bin} \operatorname{erfc}(-y_i)}{\operatorname{erfc}(y_i) + A_{bin} \operatorname{erfc}(-y_i)} \right). \quad (78)$$

Plugging γ_g from (76) and y_e from (78) into the right side of (67) and equaling it to zero one finally obtains a single equation with y_i as the only unknown. After finding y_i in this way one can reuse it to obtain γ_g through (76) and y_e through (78). Instead of doing this we will present a different path that will be more connected to what was presented in Section 2.2. We start by setting the derivative in (67) to zero to obtain

$$\begin{aligned} \gamma_g \left(\frac{1}{\operatorname{erfc}(y_e)} + \frac{1}{\operatorname{erfc}(-y_e)} \right) &= -\sqrt{\pi} \alpha e^{y_e^2} y_e + \frac{1 - \alpha}{\operatorname{erfc}(-y_e)} \\ \iff \gamma_g &= \frac{\alpha}{2} \left(-\sqrt{\pi} e^{y_e^2} y_e \operatorname{erfc}(-y_e) - 1 \right) \operatorname{erfc}(y_e) + \frac{\operatorname{erfc}(y_e)}{2}. \end{aligned} \quad (79)$$

Plugging γ_g from (76) into (77) gives

$$A_{bin} = \frac{\alpha + \frac{\alpha}{2} \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) \operatorname{erfc}(-y_i) - \operatorname{erfc}(-y_i) \right)}{1 - \alpha - \left(\frac{\alpha}{2} \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) \operatorname{erfc}(-y_i) - \operatorname{erfc}(-y_i) \right) + \beta \right)} - \frac{\frac{\alpha}{2} \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) \operatorname{erfc}(-y_i) - \operatorname{erfc}(-y_i) \right) + \beta}{\left(\frac{\alpha}{2} \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) \operatorname{erfc}(-y_i) - \operatorname{erfc}(-y_i) \right) \right)}. \quad (80)$$

After a few additional transformations we have

$$A_{bin} = \frac{2\beta + \alpha \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) - 1 \right) \operatorname{erfc}(-y_i)}{2(1 - \beta) - \alpha \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(-y_i) + 1 \right) \operatorname{erfc}(y_i)} \frac{\left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(-y_i) + 1 \right) \operatorname{erfc}(y_i)}{\left(-\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) + 1 \right) \operatorname{erfc}(-y_i)}. \quad (81)$$

A combination of (78) and (81) gives

$$\frac{\operatorname{erfc}(-y_e)}{\operatorname{erfc}(y_e)} = \frac{\operatorname{erfc}(y_i)}{\operatorname{erfc}(-y_i) A_{bin}} = \frac{2(1 - \beta) - \alpha \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(-y_i) + 1 \right) \operatorname{erfc}(y_i)}{2\beta + \alpha \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) - 1 \right) \operatorname{erfc}(-y_i)} \frac{\left(-\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) + 1 \right)}{\left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(-y_i) + 1 \right)}. \quad (82)$$

Now, from (37) and (38) we have

$$\begin{aligned} \frac{\frac{1-\beta}{\alpha} \left(\frac{2y_2 e^{y_2^2}}{\sqrt{\pi} y_2 e^{y_2^2} \operatorname{erfc}(-y_2) + 1} \right) - y_2 e^{y_2^2} \operatorname{erfc}(y_2)}{\frac{\beta}{\alpha} \left(\frac{2y_2 e^{y_2^2}}{-\sqrt{\pi} y_2 e^{y_2^2} \operatorname{erfc}(y_2) + 1} \right) - y_2 e^{y_2^2} \operatorname{erfc}(-y_2)} &= \frac{f_1(y_2; \alpha, \beta)}{-f_1(-y_2; \alpha, 1 - \beta)}, \end{aligned} \quad (83)$$

and

$$\frac{2(1 - \beta) - \alpha \left(\sqrt{\pi} y_2 e^{y_2^2} \operatorname{erfc}(-y_2) + 1 \right) \operatorname{erfc}(y_2)}{2\beta - \alpha \left(-\sqrt{\pi} y_2 e^{y_2^2} \operatorname{erfc}(y_2) + 1 \right) \operatorname{erfc}(-y_2)} \frac{\left(-\sqrt{\pi} y_2 e^{y_2^2} \operatorname{erfc}(y_2) + 1 \right)}{\left(\sqrt{\pi} y_2 e^{y_2^2} \operatorname{erfc}(-y_2) + 1 \right)} = \frac{f_1(y_2; \alpha, \beta)}{-f_1(-y_2; \alpha, 1 - \beta)}. \quad (84)$$

Connecting (82) and (84) we also have

$$\frac{\operatorname{erfc}(-y_e)}{\operatorname{erfc}(y_e)} = \frac{2(1 - \beta) - \alpha \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(-y_i) + 1 \right) \operatorname{erfc}(y_i)}{2\beta + \alpha \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) - 1 \right) \operatorname{erfc}(-y_i)} \frac{\left(-\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) + 1 \right)}{\left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(-y_i) + 1 \right)} = \frac{f_1(y_i; \alpha, \beta)}{-f_1(-y_i; \alpha, 1 - \beta)}. \quad (85)$$

Connecting further (39), (40), and (85) gives

$$\begin{aligned} \operatorname{erf}(y_e) &= \frac{f_1(y_i; \alpha, \beta) + f_1(-y_i; \alpha, 1 - \beta)}{f_1(y_i; \alpha, \beta) - f_1(-y_i; \alpha, 1 - \beta)} \\ \iff y_e &= \operatorname{erfinv} \left(\frac{f_1(y_i; \alpha, \beta) + f_1(-y_i; \alpha, 1 - \beta)}{f_1(y_i; \alpha, \beta) - f_1(-y_i; \alpha, 1 - \beta)} \right). \end{aligned} \quad (86)$$

A combination of (76) and (79)

$$\begin{aligned}\gamma_g &= \frac{\alpha}{2} \left(-\sqrt{\pi} e^{y_e^2} y_e \operatorname{erfc}(-y_e) - 1 \right) \operatorname{erfc}(y_e) + \frac{\operatorname{erfc}(y_e)}{2} = \frac{\alpha}{2} \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) \operatorname{erfc}(-y_i) - \operatorname{erfc}(-y_i) \right) + \beta \\ &= \frac{-f_1(-y_i; \alpha, 1-\beta) \alpha \left(-\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) + 1 \right)}{2e^{y_i^2} y_i}. \quad (87)\end{aligned}$$

Transforming further one also has

$$\alpha \left(-\sqrt{\pi} e^{y_e^2} y_e \operatorname{erfc}(-y_e) - 1 \right) + 1 = \frac{(f_1(y_i; \alpha, \beta) - f_1(-y_i; \alpha, 1-\beta)) \alpha \left(-\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) + 1 \right)}{2e^{y_i^2} y_i}, \quad (88)$$

and

$$e^{y_e^2} y_e \operatorname{erfc}(-y_e) = \frac{(f_1(y_i; \alpha, \beta) - f_1(-y_i; \alpha, 1-\beta)) \alpha \left(-\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) + 1 \right) - 2e^{y_i^2} y_i + 2e^{y_i^2} y_i \alpha}{-\sqrt{\pi} 2e^{y_i^2} y_i \alpha}. \quad (89)$$

Now we will argue that the right side in (89) is equal to $f_1(y_i; \alpha, \beta)$. That will follow if

$$(f_1(y_i; \alpha, \beta) - f_1(-y_i; \alpha, 1-\beta)) \alpha \left(-\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) + 1 \right) - 2e^{y_i^2} y_i + 2e^{y_i^2} y_i \alpha = -\sqrt{\pi} 2e^{y_i^2} y_i \alpha f_1(y_i; \alpha, \beta), \quad (90)$$

or

$$-f_1(-y_i; \alpha, 1-\beta) \alpha \left(-\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) + 1 \right) - 2e^{y_i^2} y_i + 2e^{y_i^2} y_i \alpha = f_1(y_i; \alpha, \beta) \alpha \left(-\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(-y_i) - 1 \right). \quad (91)$$

Utilizing (39) and (40), (91) can be rewritten in the following way

$$\frac{\alpha}{2} \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) \operatorname{erfc}(-y_i) - \operatorname{erfc}(-y_i) \right) + \beta - 1 + \alpha = - \left((1-\beta) - \left(\sqrt{\pi} y_i e^{y_i^2} \operatorname{erfc}(-y_i) + 1 \right) \frac{\alpha}{2} \operatorname{erfc}(y_i) \right). \quad (92)$$

Since (92) indeed holds one then has that (91) also holds and through (90) we have that the right side of (89) is indeed equal to $f_1(y_i; \alpha, \beta)$. Combining further (86) and (89) we finally arrive at

$$\begin{aligned}e^{y_e^2} y_e \operatorname{erfc}(-y_e) &= f_1(y_i; \alpha, \beta) \\ \iff 2 \operatorname{erfinv} \left(\frac{f_1(y_i; \alpha, \beta) + f_1(-y_i; \alpha, 1-\beta)}{f_1(y_i; \alpha, \beta) - f_1(-y_i; \alpha, 1-\beta)} \right) \frac{e^{\operatorname{erfinv} \left(\frac{f_1(y_i; \alpha, \beta) + f_1(-y_i; \alpha, 1-\beta)}{f_1(y_i; \alpha, \beta) - f_1(-y_i; \alpha, 1-\beta)} \right)^2}}{f_1(y_i; \alpha, \beta) - f_1(-y_i; \alpha, 1-\beta)} &= 1. \quad (93)\end{aligned}$$

(93) is enough to determine y_i . Then from (86) one can determine y_e and from (76) or (79) γ_g (of course, comparing (93) and (86) to (41) and (40), respectively one observes that $y_i = y_2$ and $y_e = y_1$). One can then use these values for y_i , y_e , and γ_g to determine the optimal value of $\zeta_{\alpha, \beta}(\gamma_g, y_e, y_i)$ in (58) through (59). Such a value will give $I_{err}^{(bin)}(\alpha, \beta)$ in (56). Here we will try to be a bit more explicit. Namely, we will connect the optimal $\zeta_{\alpha, \beta}(\gamma_g, y_e, y_i)$ to what we presented in Section 2.2.

2.3.5 Computing $I_{err}^{(bin)}(\alpha, \beta)$

We start by noting that $\zeta_{\alpha, \beta}(\gamma_g, y_e, y_i)$ given in (59) consists of three components, namely, Ψ_{com} , Ψ_{int} , and Ψ_{ext} given in (57). Instead of working directly with $\zeta_{\alpha, \beta}(\gamma_g, y_e, y_i)$ we will first deal with each of Ψ_{com} , Ψ_{int} , and Ψ_{ext} . From this point on we assume that y_i , y_e , and γ_g take the values determined through the

procedure explained above. Then we have for Ψ_{com}

$$\begin{aligned}
\Psi_{com} &= -(1-\beta)H\left(\frac{1-\alpha-\gamma_g}{1-\beta}\right) - \beta H\left(\frac{\beta-\gamma_g}{\beta}\right) \\
&= -(1-\alpha-\gamma_g)\log\left(\frac{1-\alpha-\gamma_g}{1-\beta}\right) - (\alpha+\gamma_g-\beta)\log\left(\frac{\alpha+\gamma_g-\beta}{1-\beta}\right) \\
&\quad - (\beta-\gamma_g)\log\left(\frac{\beta-\gamma_g}{\beta}\right) - \gamma_g\log\left(\frac{\gamma_g}{\beta}\right) \\
&= -\frac{1}{2}\left(2(1-\beta) - \alpha\left(\sqrt{\pi}e^{y_i^2}y_i\text{erfc}(-y_i) + 1\right)\text{erfc}(y_i)\right) \\
&\quad \times \log\left(\frac{\left(2(1-\beta) - \alpha\left(\sqrt{\pi}e^{y_i^2}y_i\text{erfc}(-y_i) + 1\right)\text{erfc}(y_i)\right)}{2(1-\beta)}\right) \\
&\quad - \frac{\alpha}{2}\text{erfc}(y_i)\left(\sqrt{\pi}e^{y_i^2}y_i\text{erfc}(-y_i) + 1\right)\log\left(\frac{\alpha\text{erfc}(y_i)\left(\sqrt{\pi}e^{y_i^2}y_i\text{erfc}(-y_i) + 1\right)}{2(1-\beta)}\right) \\
&\quad - \frac{\alpha}{2}\text{erfc}(-y_i)\left(-\sqrt{\pi}e^{y_i^2}y_i\text{erfc}(y_i) + 1\right)\log\left(\frac{\alpha\text{erfc}(-y_i)\left(-\sqrt{\pi}e^{y_i^2}y_i\text{erfc}(y_i) + 1\right)}{2\beta}\right) \\
&\quad - \frac{1}{2}\left(2\beta + \alpha\left(\sqrt{\pi}e^{y_i^2}y_i\text{erfc}(y_i) - 1\right)\text{erfc}(-y_i)\right)\log\left(\frac{2\beta + \alpha\left(\sqrt{\pi}e^{y_i^2}y_i\text{erfc}(y_i) - 1\right)\text{erfc}(-y_i)}{2\beta}\right).
\end{aligned} \tag{94}$$

Similarly we have for Ψ_{int}

$$\begin{aligned}
\Psi_{int} &= \alpha y_i^2 + (\alpha - \beta + \gamma_g)\log(\text{erfc}(y_i)) + (\beta - \gamma_g)\log(\text{erfc}(-y_i)) - \alpha\log(2) \\
&= \alpha y_i^2 + \frac{\alpha}{2}\text{erfc}(y_i)\left(\sqrt{\pi}e^{y_i^2}y_i\text{erfc}(-y_i) + 1\right)\log(\text{erfc}(y_i)) \\
&\quad + \frac{\alpha}{2}\text{erfc}(-y_i)\left(-\sqrt{\pi}e^{y_i^2}y_i\text{erfc}(y_i) + 1\right)\log(\text{erfc}(-y_i)) - \alpha\log(2),
\end{aligned} \tag{95}$$

and finally for Ψ_{ext}

$$\begin{aligned}
\Psi_{ext} &= -\alpha y_e^2 + (1-\alpha-\gamma_g)\log(\text{erfc}(-y_e)) + \gamma_g\log(\text{erfc}(y_e)) - (1-\alpha)\log(2) \\
&= -\alpha y_e^2 + \frac{1}{2}\left(2(1-\beta) - \alpha\left(\sqrt{\pi}e^{y_i^2}y_i\text{erfc}(-y_i) + 1\right)\text{erfc}(y_i)\right)\log(\text{erfc}(-y_e)) \\
&\quad + \frac{1}{2}\left(2\beta + \alpha\left(\sqrt{\pi}e^{y_i^2}y_i\text{erfc}(y_i) - 1\right)\text{erfc}(-y_i)\right)\log(\text{erfc}(y_e)) - (1-\alpha)\log(2).
\end{aligned} \tag{96}$$

Combining (94), (95), and (96) we also have

$$\begin{aligned}
\Psi_{net}^{(bin)} &= -\frac{1}{2}\left(2(1-\beta) - \alpha\left(\sqrt{\pi}e^{y_i^2}y_i\text{erfc}(-y_i) + 1\right)\text{erfc}(y_i)\right) \\
&\quad \times \log\left(\frac{\left(2(1-\beta) - \alpha\left(\sqrt{\pi}e^{y_i^2}y_i\text{erfc}(-y_i) + 1\right)\text{erfc}(y_i)\right)}{2(1-\beta)\text{erfc}(-y_e)}\right) \\
&\quad - \frac{\alpha}{2}\text{erfc}(y_i)\left(\sqrt{\pi}e^{y_i^2}y_i\text{erfc}(-y_i) + 1\right)\log\left(\frac{\alpha\left(\sqrt{\pi}e^{y_i^2}y_i\text{erfc}(-y_i) + 1\right)}{2(1-\beta)}\right)
\end{aligned}$$

$$\begin{aligned}
& -\frac{\alpha}{2} \operatorname{erfc}(-y_i) \left(-\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) + 1 \right) \log \left(\frac{\alpha \left(-\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) + 1 \right)}{2\beta} \right) \\
& -\frac{1}{2} \left(2\beta + \alpha \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) - 1 \right) \operatorname{erfc}(-y_i) \right) \log \left(\frac{2\beta + \alpha \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) - 1 \right) \operatorname{erfc}(-y_i)}{2\beta \operatorname{erfc}(y_e)} \right) \\
& + \alpha y_i^2 - \alpha y_e^2 - \log(2).
\end{aligned} \tag{97}$$

Utilizing again (39) and (40), (97) can be further transformed

$$\begin{aligned}
\Psi_{net}^{(bin)} &= -\frac{1}{2} \left(2(1 - \beta) - \alpha \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(-y_i) + 1 \right) \operatorname{erfc}(y_i) \right) \\
& \times \log \left(\frac{y_e e^{y_e^2} \operatorname{erfc}(-y_e) \alpha \left(\sqrt{\pi} y_i e^{y_i^2} \operatorname{erfc}(-y_i) + 1 \right)}{2(1 - \beta) \operatorname{erfc}(-y_e) y_i e^{y_i^2}} \right) \\
& -\frac{\alpha}{2} \operatorname{erfc}(y_i) \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(-y_i) + 1 \right) \log \left(\frac{\alpha \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(-y_i) + 1 \right)}{2(1 - \beta)} \right) \\
& -\frac{\alpha}{2} \operatorname{erfc}(-y_i) \left(-\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) + 1 \right) \log \left(\frac{\alpha \left(-\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) + 1 \right)}{2\beta} \right) \\
& -\frac{1}{2} \left(2\beta + \alpha \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) - 1 \right) \operatorname{erfc}(-y_i) \right) \log \left(\frac{y_e e^{y_e^2} \operatorname{erfc}(y_e) \alpha \left(-\sqrt{\pi} y_i e^{y_i^2} \operatorname{erfc}(y_i) + 1 \right)}{2\beta \operatorname{erfc}(y_e) y_i e^{y_i^2}} \right) \\
& + \alpha y_i^2 - \alpha y_e^2 - \log(2).
\end{aligned} \tag{98}$$

Continuing further we also obtain

$$\begin{aligned}
\Psi_{net}^{(bin)} &= -\frac{1}{2} \left(2(1 - \beta) - \alpha \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(-y_i) + 1 \right) \operatorname{erfc}(y_i) \right) \log \left(\frac{y_e e^{y_e^2}}{y_i e^{y_i^2}} \right) \\
& -\frac{1}{2} \left(2\beta + \alpha \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) - 1 \right) \operatorname{erfc}(-y_i) \right) \log \left(\frac{y_e e^{y_e^2}}{y_i e^{y_i^2}} \right) \\
& -\frac{1}{2} \left(2(1 - \beta) - \alpha \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(-y_i) + 1 \right) \operatorname{erfc}(y_i) \right) \log \left(\frac{\alpha \left(\sqrt{\pi} y_i e^{y_i^2} \operatorname{erfc}(-y_i) + 1 \right)}{2(1 - \beta)} \right) \\
& -\frac{1}{2} \left(2\beta + \alpha \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) - 1 \right) \operatorname{erfc}(-y_i) \right) \log \left(\frac{\alpha \left(-\sqrt{\pi} y_i e^{y_i^2} \operatorname{erfc}(y_i) + 1 \right)}{2\beta} \right) \\
& -\frac{\alpha}{2} \operatorname{erfc}(y_i) \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(-y_i) + 1 \right) \log \left(\frac{\alpha \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(-y_i) + 1 \right)}{2(1 - \beta)} \right) \\
& -\frac{\alpha}{2} \operatorname{erfc}(-y_i) \left(-\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) + 1 \right) \log \left(\frac{\alpha \left(-\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) + 1 \right)}{2\beta} \right) \\
& + \alpha y_i^2 - \alpha y_e^2 - \log(2),
\end{aligned} \tag{99}$$

and

$$\begin{aligned}\Psi_{net}^{(bin)} &= -(1-\alpha) \log \left(\frac{y_e e^{y_e^2}}{y_i e^{y_i^2}} \right) + (1-\beta) \log \left(\frac{2(1-\beta)}{\alpha (\sqrt{\pi} y_i e^{y_i^2} \operatorname{erfc}(-y_i) + 1)} \right) \\ &\quad + \beta \log \left(\frac{2\beta}{\alpha (-\sqrt{\pi} y_i e^{y_i^2} \operatorname{erfc}(y_i) + 1)} \right) + \alpha y_i^2 - \alpha y_e^2 - \log(2).\end{aligned}\quad (100)$$

A couple of additional cosmetic changes finally give

$$\begin{aligned}\Psi_{net}^{(bin)} = I_{err}^{(bin)}(\alpha, \beta) &= (\alpha - 1) \log \left(\frac{y_e}{y_i} \right) + (1-\beta) \log \left(\frac{1-\beta}{\alpha (\sqrt{\pi} y_i e^{y_i^2} \operatorname{erfc}(-y_i) + 1)} \right) \\ &\quad + \beta \log \left(\frac{\beta}{\alpha (-\sqrt{\pi} y_i e^{y_i^2} \operatorname{erfc}(y_i) + 1)} \right) + y_i^2 - y_e^2 = I_{ldp}^{(bin)}(\alpha, \beta).\end{aligned}\quad (101)$$

Comparing (44) and (101) one recognizes that $I_{err}^{(bin)} = I_{ldp}^{(bin)} = I_{err,u}^{(bin,ub)}$ (of course $y_e \leftrightarrow y_1$ and $y_i \leftrightarrow y_2$) and observes that the choice for ν , A_0 , c_3 , and γ made in (42) is indeed optimal. Moreover, in the lower tail regime ($\alpha < \alpha_w$, where α_w is such that $\psi_\beta^{(bin)}(\alpha_w) = \xi_{\alpha_w}^{(bin)}(\beta) = 1$) considerations from [15] ensure that one also has

$$\Psi_{net}^{(bin)}(\alpha, \beta) = I_{cor}^{(bin)}(\alpha, \beta) \triangleq \lim_{n \rightarrow \infty} \frac{\log P_{cor}}{n} = \Psi_{com} + \Psi_{int} - \Psi_{ext}, \quad (102)$$

where Ψ_{com} , Ψ_{int} , and Ψ_{ext} are as in (57). The above considerations then automatically fully characterize the binary ℓ_1 's LDP. The characterization is summarized in the following theorem.

Theorem 8 (Binary ℓ_1 's LDP). *Assume the setup of Theorem 1 and assume that a pair (α, β) is given. Let P_{err} be the probability that the solutions of (2) and (4) coincide and let P_{cor} be the probability that the solutions of (2) and (4) do not coincide. Set*

$$f_1(y_2; \alpha, \beta) \triangleq \frac{1-\beta}{\alpha} \left(\frac{2y_2 e^{y_2^2}}{\sqrt{\pi} y_2 e^{y_2^2} \operatorname{erfc}(-y_2) + 1} \right) - y_2 e^{y_2^2} \operatorname{erfc}(y_2). \quad (103)$$

Also let y_2 and y_1 satisfy the following **fundamental characterizations of the binary ℓ_1 's LDP** and achieve the optimum in (58):

$$2 \operatorname{erf} \operatorname{inv} \left(\frac{f_1(y_2; \alpha, \beta) + f_1(-y_2; \alpha, 1-\beta)}{f_1(y_2; \alpha, \beta) - f_1(-y_2; \alpha, 1-\beta)} \right) \frac{e^{\operatorname{erf} \operatorname{inv} \left(\frac{f_1(y_2; \alpha, \beta) + f_1(-y_2; \alpha, 1-\beta)}{f_1(y_2; \alpha, \beta) - f_1(-y_2; \alpha, 1-\beta)} \right)^2}}{f_1(y_2; \alpha, \beta) - f_1(-y_2; \alpha, 1-\beta)} = 1. \quad (104)$$

and

$$y_1 = \operatorname{erf} \operatorname{inv} \left(\frac{f_1(y_2; \alpha, \beta) + f_1(-y_2; \alpha, 1-\beta)}{f_1(y_2; \alpha, \beta) - f_1(-y_2; \alpha, 1-\beta)} \right). \quad (105)$$

Finally, let $I_{ldp}^{(bin)}(\alpha, \beta)$ be defined through the following **binary ℓ_1 's fundamental LDP rate function** characterization

$$\begin{aligned}I_{ldp}^{(bin)}(\alpha, \beta) &\triangleq (\alpha - 1) \log \left(\frac{y_1}{y_2} \right) + (1-\beta) \log \left(\frac{1-\beta}{\alpha (\sqrt{\pi} e^{y_2^2} y_2 \operatorname{erfc}(-y_2) + 1)} \right) \\ &\quad + \beta \log \left(\frac{\beta}{\alpha (-\sqrt{\pi} e^{y_2^2} y_2 \operatorname{erfc}(y_2) + 1)} \right) + y_2^2 - y_1^2.\end{aligned}\quad (106)$$

Then if $\alpha > \alpha_w$

$$I_{err}^{(bin)}(\alpha, \beta) \triangleq \lim_{n \rightarrow \infty} \frac{\log P_{err}}{n} = I_{ldp}^{(bin)}(\alpha, \beta). \quad (107)$$

Moreover, if $\alpha < \alpha_w$

$$I_{cor}^{(bin)}(\alpha, \beta) \triangleq \lim_{n \rightarrow \infty} \frac{\log P_{cor}}{n} = I_{ldp}^{(bin)}(\alpha, \beta). \quad (108)$$

Proof. Follows from the above discussion. \square

2.3.6 Reestablishing the phase transitions

In this section we will show how one can reestablish the phase transition results from Section 2.1 utilizing Theorem 8 and the above considerations leading up to Theorem 8. We start by focusing at pairs (α, β) for which $y_1 = y_2$ in Theorem 8 (of course, we may not know a priori if such pairs do exist; nonetheless, we will make such an assumption and since we will not contradict it through the derivation below, it will follow that the assumption is actually correct). From (37) and (38) we have

$$\begin{aligned} y_2 e^{y_2^2} \operatorname{erfc}(-y_2) &= y_1 e^{y_1^2} \operatorname{erfc}(-y_1) = \frac{1-\beta}{\alpha} \left(\frac{2y_2 e^{y_2^2}}{\sqrt{\pi} y_2 e^{y_2^2} \operatorname{erfc}(-y_2) + 1} \right) - y_2 e^{y_2^2} \operatorname{erfc}(y_2) = f_1(y_2; \alpha, \beta) \\ y_2 e^{y_2^2} \operatorname{erfc}(y_2) &= y_1 e^{y_1^2} \operatorname{erfc}(y_1) = \frac{\beta}{\alpha} \left(\frac{2y_2 e^{y_2^2}}{-\sqrt{\pi} y_2 e^{y_2^2} \operatorname{erfc}(y_2) + 1} \right) - y_2 e^{y_2^2} \operatorname{erfc}(-y_2) = -f_1(-y_2; \alpha, 1-\beta). \end{aligned} \quad (109)$$

A couple of simple algebraic operations transform (109) into the following

$$\begin{aligned} 1 &= \frac{1-\beta}{\alpha} \left(\frac{1}{\sqrt{\pi} y_2 e^{y_2^2} \operatorname{erfc}(-y_2) + 1} \right) \\ 1 &= \frac{\beta}{\alpha} \left(\frac{1}{-\sqrt{\pi} y_2 e^{y_2^2} \operatorname{erfc}(y_2) + 1} \right). \end{aligned} \quad (110)$$

Transforming a bit further we also have

$$\begin{aligned} \sqrt{\pi} y_2 e^{y_2^2} \operatorname{erfc}(-y_2) &= \frac{1-\beta}{\alpha} - 1 \\ \sqrt{\pi} y_2 e^{y_2^2} \operatorname{erfc}(y_2) &= 1 - \frac{\beta}{\alpha}, \end{aligned} \quad (111)$$

and

$$\frac{1 + \operatorname{erf}(y_2)}{1 - \operatorname{erf}(y_2)} = \frac{\operatorname{erfc}(-y_2)}{\operatorname{erfc}(y_2)} = \frac{1 - \alpha - \beta}{\alpha - \beta}. \quad (112)$$

From (112) we then easily have

$$\begin{aligned} \operatorname{erf}(y_2) &= \frac{\frac{1-\alpha-\beta}{\alpha-\beta} - 1}{\frac{1-\alpha-\beta}{\alpha-\beta} + 1} = \frac{1-2\alpha}{1-2\beta} \\ \iff y_2 &= \operatorname{erfinv} \left(\frac{1-2\alpha}{1-2\beta} \right). \end{aligned} \quad (113)$$

A combination of (111) and (113) then gives

$$\sqrt{\pi} y_2 e^{y_2^2} \operatorname{erfc}(-y_2) = \sqrt{\pi} y_2 e^{y_2^2} (1 + \operatorname{erf}(y_2)) = \sqrt{\pi} y_2 e^{y_2^2} \frac{2 - 2\beta - 2\alpha}{1 - 2\beta} = \frac{1 - \beta - \alpha}{\alpha}. \quad (114)$$

Finally combining (113) and (114) we obtain

$$\frac{(1-2\beta)e^{-y_2^2}}{2\sqrt{\pi}\alpha y_2} = \frac{(1-2\beta)e^{-(\operatorname{erf}^{-1}(\frac{1-2\alpha}{1-2\beta}))^2}}{2\sqrt{\pi}\alpha \operatorname{erf}^{-1}(\frac{1-2\alpha}{1-2\beta})} = 1. \quad (115)$$

It is not that hard to see that (115) is exactly the same as (5). Moreover, from (109) we also have

$$f_1(y_2; \alpha, \beta) - f_1(-y_2; \alpha, 1-\beta) = y_2 e^{y_2^2} \operatorname{erfc}(-y_2) + y_2 e^{y_2^2} \operatorname{erfc}(y_2) = 2y_2 e^{y_2^2}, \quad (116)$$

and

$$f_1(y_2; \alpha, \beta) + f_1(-y_2; \alpha, 1-\beta) = y_2 e^{y_2^2} \operatorname{erfc}(-y_2) - y_2 e^{y_2^2} \operatorname{erfc}(y_2) = 2y_2 e^{y_2^2} \operatorname{erf}(y_2). \quad (117)$$

Now, if (α, β) are such that (115) holds then $y_1 = y_2$ and (116) and (117) ensure that (104) and (105) hold and that $I_{ldp}^{(bin)}(\alpha, \beta) = 0$ in (106) which is exactly the value that $I_{ldp}^{(bin)}(\alpha, \beta)$ takes at the phase transition.

2.4 Theoretical and numerical LDP results

In this section we present in a bit more concrete way what is actually proven in Theorem 8. The theoretical LDP rate function curve that can be obtained based on Theorem 8 is shown in Figure 2. Two different values for β were selected, $\beta = 0.22933$ (which can be obtained from the PT curve for $\alpha = 0.4$) and $\beta = \frac{1}{3}$. Also, for $\beta = 0.22933$, we in addition to Figure 2 provide Table 1 that contains the numerical values for all the quantities of interest in Theorems 5 and 8. Finally, in Figure 3 and Table 2 we show how the simulated values compare to the theoretical ones. As can be observed, even for fairly small dimensions (of order 100) one already approaches the theoretical curves (the theoretical curves are of course derived for an infinite dimensional asymptotic regime).

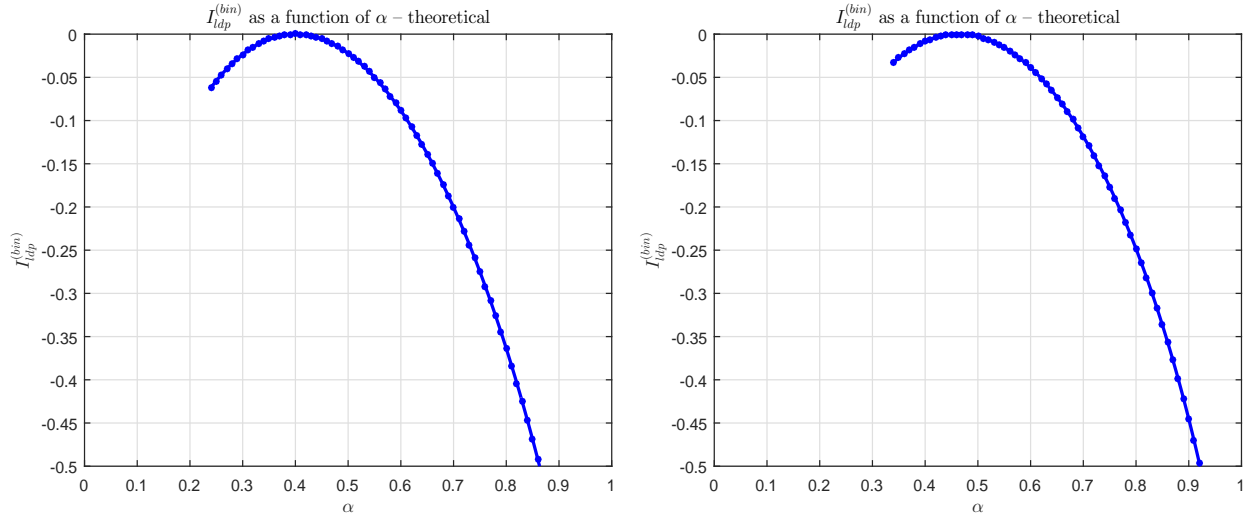


Figure 2: $I_{ldp}^{(bin)}$ as a function of α ; left – $\beta = 0.22933$; right – $\beta = \frac{1}{3}$

3 Box ℓ_1

In Section 2 we looked at the properties of the ℓ_1 from (4) when employed for solving (2) a priori known to have the solution that is a binary vector. Here we will instead of binary vectors consider the so-called box-constrained vectors (for more on these see, e.g. [7]). These vectors are typically viewed as a more general class of binary vectors. Namely, any component of a box-constrained vector is assumed to be within a real

Table 1: A collection of values for y_1 , γ_g , A_{bin} , y_2 , ν , A_0 , c_3 , γ , and $I_{ldp}^{(bin)}$ in Theorem 5; $\beta = 0.22933$

α	0.30	0.35	0.40	0.45	0.50
y_1	0.3923	0.3666	0.3401	0.3127	0.2846
γ_g	0.1026	0.0866	0.0723	0.0596	0.0484
A_{bin}	0.2347	0.2234	0.2120	0.2006	0.1892
y_2	0.2431	0.2907	0.3401	0.3910	0.4432
ν	0.5548	0.5185	0.4810	0.4422	0.4025
A_0	1.6137	1.2611	1.0000	0.7997	0.6421
c_3	-0.5445	-0.2770	0.0000	0.3023	0.6471
γ	0.1697	0.2346	0.3162	0.4194	0.5506
$I_{ldp}^{(bin)}$	-0.0234	-0.0058	0.0000	-0.0056	-0.0223

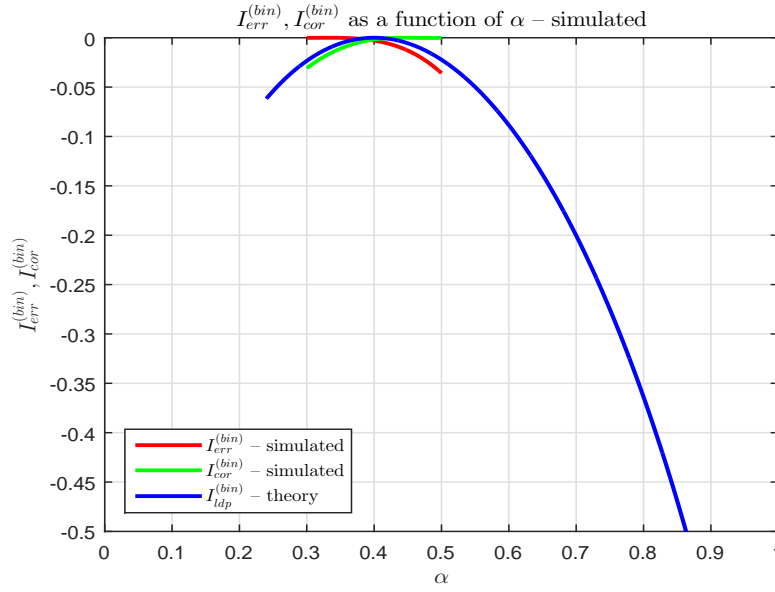


Figure 3: Binary ℓ_1 's weak LDP rate function – theory and simulation; $\beta = 0.22933$

interval (as earlier, without the loss of generality, we will assume that this interval is $[0, 1]$). As discussed in the introduction, this broadly defines pretty much any vector. Things are a bit more interesting if one looks at the so-called sparse box-constrained vectors. Such vectors are defined in the following way. Namely, vector \mathbf{x} is called box-constrained k sparse (from this point on we assume being constrained on interval $[0, 1]$) if no more than k of its components are not at the edges of the box/interval. In other words, \mathbf{x} is called box-constrained k sparse if no more than k of its components are not equal to zero or one. To solve (2) known to have box-constrained k sparse solution we will employ (4) which we conveniently rewrite below

$$\begin{aligned}
& \min && \|\mathbf{x}\|_1 \\
& \text{subject to} && A\mathbf{x} = \mathbf{y} \\
& && 0 \leq \mathbf{x}_i \leq 1, 1 \leq i \leq n.
\end{aligned} \tag{118}$$

As mentioned earlier, we will refer to (118) as the *box* ℓ_1 when it is used for finding the box-constrained k sparse solution of (2). Below we will provide a performance analysis of the *box* ℓ_1 . As was the case in Section 2 when we discussed the *binary* ℓ_1 , the analysis will focus on the phase transitions (PTs) and the corresponding LDPs. Differently from what we did in Section 2 though, here we start things off by focusing

Table 2: $I_{err}^{(bin)}$, $I_{cor}^{(bin)}$ – simulated; $I_{ldp}^{(bin)}$ calculated for $\beta = 0.22933$

α	0.30	0.35	0.40	0.45	0.50
k	32	69	69	69	32
m	42	105	120	135	70
n	140	300	300	300	140
$I_{err}^{(bin)}$ – simulated	–0.0001	–0.0001	–0.0029	–0.0130	–0.0359
$I_{cor}^{(bin)}$ – simulated	–0.0309	–0.0110	–0.0018	–0.0001	–0.0000
$I_{ldp}^{(bin)}$ – theory	–0.0234	–0.0058	0.0000	–0.0056	–0.0223

on the LDPs first (the PT results will easily follow afterwards).

3.1 Large deviations

We will try to follow as closely as possible the derivations for the large deviations of the binary ℓ_1 that we presented in Section 2. As usual, we will try to skip as many repetitive steps as possible and instead will focus on those that bring key differences. We begin by introducing a theorem that is basically the box ℓ_1 analogue to the binary ℓ_1 's Theorem 2. To facilitate the exposition, in addition to assuming that all elements of \mathbf{x} are from $[0, 1]$ interval, we will, for any $\mu \in [\frac{1}{2}, 1]$, without the loss of generality assume that the elements $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{\mu(n-k)}$ of \mathbf{x} are equal to zero and that the elements $\mathbf{x}_{\mu(n-k)+1}, \mathbf{x}_{\mu(n-k)+2}, \dots, \mathbf{x}_{n-k}$ of \mathbf{x} are equal to one. Minimal modifications of the arguments leading up to Theorem 2 produce the following theorem.

Theorem 9. (*[23–25] Nonzero elements of box-constrained \mathbf{x} have fixed location*) Assume that an $m \times n$ system matrix A is given. Let \mathbf{x} be a box-constrained k -sparse vector and let μ be a real number such that $\mu \in [\frac{1}{2}, 1]$. Also let each element of \mathbf{x} belong to $[0, 1]$ interval and let $\mathbf{x}_1 = \mathbf{x}_2 = \dots = \mathbf{x}_{\mu(n-k)} = 0$ and $\mathbf{x}_{\mu(n-k)+1}, \mathbf{x}_{\mu(n-k)+2}, \dots, \mathbf{x}_{n-k} = 1$. Further, assume that $\mathbf{y} \triangleq A\mathbf{x}$ and that \mathbf{w} is an $n \times 1$ vector such that $\mathbf{w}_i \geq 0, 1 \leq i \leq \mu(n-k)$, and $\mathbf{w}_i \leq 0, \mu(n-k)+1 \leq i \leq n-k$. If

$$(\forall \mathbf{w} \in \mathbf{R}^n | A\mathbf{w} = 0) \quad - \quad \sum_{i=n-k+1}^n \mathbf{w}_i < \sum_{i=1}^{n-k} \mathbf{w}_i, \quad (119)$$

then the solutions of (2) and (118) coincide. Moreover, if

$$(\exists \mathbf{w} \in \mathbf{R}^n | A\mathbf{w} = 0) \quad - \quad \sum_{i=n-k+1}^n \mathbf{w}_i \geq \sum_{i=1}^{n-k} \mathbf{w}_i, \quad (120)$$

then the solution of (2) is not the solution of (118).

Proof. Follows by a couple of simple modifications of the arguments leading up to Theorem 2. \square

To facilitate the exposition we set

$$S_w^{(box)} \triangleq \{\mathbf{w} \in S^{n-1} | \quad - \quad \sum_{i=n-k+1}^n \mathbf{w}_i < \sum_{i=1}^{n-k} \mathbf{w}_i, \quad \mathbf{w}_i \geq 0, 1 \leq i \leq \mu(n-k), \quad \text{and} \quad \mathbf{w}_i \leq 0, \mu(n-k)+1 \leq i \leq n-k\}, \quad (121)$$

and as in Section 2 (and ultimately [17]), we first provide a detailed analysis of the LDPs upper tail (also as in Section 2, a few minimal adaptations of the upper tail analysis automatically settle the lower tail as well).

3.1.1 Upper tail

We recall that in the upper tail we consider the points (α, β) such that $\alpha \geq \alpha_w$ where α_w is the phase transition value for given β . Assuming that the elements of A are i.i.d. standard normals and following Section 2 and [17], we have

$$P_{err}^{(box)} \triangleq P\left(\min_{\mathbf{w} \in S_w^{(box)}} \|\mathbf{A}\mathbf{w}\|_2 \leq 0\right) = P\left(\max_{\mathbf{w} \in S_w^{(box)}} \min_{\|\mathbf{y}\|_2=1} (\mathbf{y}^T \mathbf{A}\mathbf{w}) \geq 0\right) \leq \min_{c_3 \geq 0} e^{-\frac{c_3^2}{2}} E e^{-c_3 \|\mathbf{g}\|_2} E e^{c_3 w(\mathbf{h}, S_w^{(box)})}, \quad (122)$$

where $P_{err}^{(box)}$ is the so-called probability of error/failure, i.e. the probability that (118) fails to produce the solution of (2) and

$$\begin{aligned} w(\mathbf{h}, S_w^{(box)}) &\triangleq \max_{\mathbf{w} \in S_w^{(box)}} (\mathbf{h}^T \mathbf{w}) = \max_{\bar{\mathbf{y}} \in \mathbb{R}^n} \sum_{i=1}^n \mathbf{h}_i \bar{\mathbf{y}}_i \\ &\text{subject to} \quad \bar{\mathbf{y}}_i \geq 0, 0 \leq i \leq n-k \\ &\quad \sum_{i=n-k+1}^n \bar{\mathbf{y}}_i \geq \sum_{i=1}^{\mu(n-k)} \bar{\mathbf{y}}_i - \sum_{i=\mu(n-k)+1}^{n-k} \bar{\mathbf{y}}_i \\ &\quad \sum_{i=1}^n \bar{\mathbf{y}}_i^2 \leq 1. \end{aligned} \quad (123)$$

We of course recall that the elements of \mathbf{h} are again the i.i.d. standard normals. As in [14, 16–18, 20, 23, 25, 26] one writes

$$\begin{aligned} w(\mathbf{h}, S_w^{(box)}) &= - \max_{\nu \geq 0, \gamma \geq 0} \min_{\bar{\mathbf{y}}} \sum_{i=1}^n -\mathbf{h}_i \bar{\mathbf{y}}_i + \nu \sum_{i=1}^{\mu(n-k)} \bar{\mathbf{y}}_i - \nu \sum_{i=\mu(n-k)+1}^{n-k} \bar{\mathbf{y}}_i - \nu \sum_{i=n-k+1}^n \bar{\mathbf{y}}_i + \gamma \sum_{i=1}^n \bar{\mathbf{y}}_i^2 - \gamma \\ &\text{subject to} \quad \bar{\mathbf{y}}_i \geq 0, 0 \leq i \leq n-k, \end{aligned} \quad (124)$$

and finally

$$\begin{aligned} w(\mathbf{h}, S_w^{(box)}) &= \min_{\nu \geq 0, \gamma \geq 0} \frac{\sum_{i=1}^{\mu(n-k)} \max(\mathbf{h}_i - \nu, 0)^2 + \sum_{i=\mu(n-k)+1}^{n-k} \max(\mathbf{h}_i + \nu, 0)^2 + \sum_{i=n-k+1}^n (\mathbf{h}_i + \nu)^2}{4\gamma} + \gamma \\ &= \min_{\nu \geq 0} \sqrt{\sum_{i=1}^{\mu(n-k)} \max(\mathbf{h}_i - \nu, 0)^2 + \sum_{i=\mu(n-k)+1}^{n-k} \max(\mathbf{h}_i + \nu, 0)^2 + \sum_{i=n-k+1}^n (\mathbf{h}_i + \nu)^2}. \end{aligned} \quad (125)$$

The following theorem summarizes the above methodology to upper bound $P_{err}^{(box)}$.

Theorem 10. *Let A be an $m \times n$ matrix in (2) with i.i.d. standard normal components. Let μ be a real number such that $\mu \in [\frac{1}{2}, 1]$. Further, let the unknown \mathbf{x} in (2) be box-constrained k -sparse and let the locations of the elements of \mathbf{x} from $(0, 1)$ be arbitrarily chosen but fixed. Let $P_{err}^{(box)}$ be the probability that the solution of (118) is not the solution of (2). Then*

$$\begin{aligned} P_{err}^{(box)} &\leq \min_{c_3 \geq 0} e^{-\frac{c_3^2}{2}} e^{-c_3 \|\mathbf{g}\|_2} E e^{c_3 w(\mathbf{h}, S_w^{(box)})} \\ &= \min_{c_3 \geq 0} \left(e^{-\frac{c_3^2}{2}} \frac{1}{\sqrt{2\pi}^m} \int_{\mathbf{g}} e^{-\sum_{i=1}^m \mathbf{g}_i^2 / 2 - c_3 \|\mathbf{g}\|_2} d\mathbf{g} \min_{\nu \geq 0, \gamma \geq \frac{c_3}{2}} w_1^{\mu(n-k)} w_2^{(1-\mu)(n-k)} w_3^k e^{c_3 \gamma} \right), \end{aligned} \quad (126)$$

where

$$\begin{aligned}
w_1 &= \frac{1}{\sqrt{2\pi}} \int_h e^{-h^2/2} e^{c_3 \max(h-\nu, 0)^2/4/\gamma} dh = \frac{1}{2} \left(\frac{e^{\frac{c_3 \nu^2/4/\gamma}{1-c_3/2/\gamma}}}{\sqrt{1-c_3/2/\gamma}} \operatorname{erfc} \left(\frac{\nu}{\sqrt{2}\sqrt{1-c_3/2/\gamma}} \right) + \operatorname{erf} \left(\frac{\nu}{\sqrt{2}} \right) + 1 \right) \\
w_2 &= \frac{1}{\sqrt{2\pi}} \int_h e^{-h^2/2} e^{c_3 \max(h+\nu, 0)^2/4/\gamma} dh = \frac{1}{2} \left(\frac{e^{\frac{c_3 \nu^2/4/\gamma}{1-c_3/2/\gamma}}}{\sqrt{1-c_3/2/\gamma}} \operatorname{erfc} \left(\frac{-\nu}{\sqrt{2}\sqrt{1-c_3/2/\gamma}} \right) + \operatorname{erf} \left(\frac{-\nu}{\sqrt{2}} \right) + 1 \right) \\
w_3 &= \frac{1}{\sqrt{2\pi}} \int_h e^{-h^2/2} e^{c_3 (h+\nu)^2/4/\gamma} dh = \frac{e^{\frac{c_3 \nu^2/4/\gamma}{1-c_3/2/\gamma}}}{\sqrt{1-c_3/2/\gamma}}.
\end{aligned} \tag{127}$$

Proof. Follows from the above considerations, what was presented in Section 2, and ultimately through the mechanisms developed in [14, 16–18, 20, 23, 25, 26]. \square

The above upper-bounding strategy works for any allowable integers m , k , and n . Introducing for the decay rate of $P_{err}^{(box)}$, $I_{err}^{(box)}(\alpha, \beta)$,

$$I_{err}^{(box)}(\alpha, \beta) \triangleq \lim_{n \rightarrow \infty} \frac{\log P_{err}^{(box)}}{n}. \tag{128}$$

we have, based on Theorem 10, the following LDP type of theorem.

Theorem 11. Assume the setup of Theorem 10. Further, let integers m , k , and n be large ($k \leq m \leq n$) such that $\beta = \frac{k}{n}$ and $\alpha = \frac{m}{n}$ are constants independent of n . Assume that a pair (α, β) is given. Also, assume the following scaling: $c_3 \rightarrow c_3 \sqrt{n}$ and $\gamma \rightarrow \gamma \sqrt{n}$. Then

$$\begin{aligned}
I_{err}^{(box)}(\alpha, \beta) &\triangleq \lim_{n \rightarrow \infty} \frac{\log P_{err}^{(box)}}{n} \\
&\leq \min_{c_3 \geq 0} \left(-\frac{(c_3)^2}{2} + I_{sph} + \min_{\nu \geq 0, \gamma \geq \frac{c_3}{2}} (\mu(1-\beta) \log w_1 + (1-\mu)(1-\beta) \log w_2 + \beta \log w_3 + c_3 \gamma) \right) \\
&\triangleq I_{err,u}^{(box,ub)}(\alpha, \beta),
\end{aligned} \tag{129}$$

where

$$\begin{aligned}
I_{sph} &= \hat{\gamma} c_3 - \frac{\alpha}{2} \log \left(1 - \frac{c_3}{2\hat{\gamma}} \right) \\
\hat{\gamma} &= \frac{c_3 - \sqrt{(c_3)^2 + 4\alpha}}{4} \\
w_1 &= \frac{1}{2} \left(\frac{e^{\frac{c_3 \nu^2/4/\gamma}{1-c_3/2/\gamma}}}{\sqrt{1-c_3/2/\gamma}} \operatorname{erfc} \left(\frac{\nu}{\sqrt{2}\sqrt{1-c_3/2/\gamma}} \right) + \operatorname{erf} \left(\frac{\nu}{\sqrt{2}} \right) + 1 \right) \\
w_2 &= \frac{1}{2} \left(\frac{e^{\frac{c_3 \nu^2/4/\gamma}{1-c_3/2/\gamma}}}{\sqrt{1-c_3/2/\gamma}} \operatorname{erfc} \left(\frac{-\nu}{\sqrt{2}\sqrt{1-c_3/2/\gamma}} \right) + \operatorname{erf} \left(\frac{-\nu}{\sqrt{2}} \right) + 1 \right) \\
w_3 &= \frac{e^{\frac{c_3 \nu^2/4/\gamma}{1-c_3/2/\gamma}}}{\sqrt{1-c_3/2/\gamma}}.
\end{aligned} \tag{130}$$

Proof. As in the case of Theorem 4 in Section 2, follows in a fashion analogous to the one employed in [17]. \square

Below we will provide an explicit solution to the above optimization problem. We will follow the methodology of Section 2 as closely as possible. However, there will be quite a few differences and we will try to

emphasize them.

3.1.2 Determining $I_{err,u}^{(box,ub)}$

As in Section 2 we start by setting

$$A_0 \triangleq \sqrt{1 - \frac{c_3}{2\gamma}}, \quad (131)$$

and then note that optimization problem from the above theorem can be rewritten as

$$I_{err,u}^{(box,ub)}(\alpha, \beta) \triangleq \min_{c_3 \geq 0, \nu \geq 0, A_0 \leq 1} \zeta_{\alpha,\beta}^{(box)}(c_3, \nu, A_0), \quad (132)$$

where

$$\begin{aligned} \zeta_{\alpha,\beta}^{(box)}(c_3, \nu, A_0) &= \left(-\frac{c_3^2}{2} + I_{sph} + \mu(1 - \beta) \log w_1 + (1 - \mu)(1 - \beta) \log w_2 + \beta \log w_3 + \frac{c_3^2}{2(1 - A_0^2)} \right) \\ I_{sph} &= \hat{\gamma} c_3 - \frac{\alpha}{2} \log \left(1 - \frac{c_3}{2\hat{\gamma}} \right) \\ \hat{\gamma} &= \frac{c_3 - \sqrt{(c_3)^2 + 4\alpha}}{4} \\ w_1 &= \frac{1}{2} \left(\frac{e^{\frac{(1-A_0^2)\nu^2}{2A_0^2}}}{A_0} \operatorname{erfc} \left(\frac{\nu}{\sqrt{2}A_0} \right) + \operatorname{erf} \left(\frac{\nu}{\sqrt{2}} \right) + 1 \right) \\ w_2 &= \frac{1}{2} \left(\frac{e^{\frac{(1-A_0^2)\nu^2}{2A_0^2}}}{A_0} \operatorname{erfc} \left(\frac{-\nu}{\sqrt{2}A_0} \right) + \operatorname{erf} \left(\frac{-\nu}{\sqrt{2}} \right) + 1 \right) \\ w_3 &= \frac{e^{\frac{c_3\nu^2/4/\gamma}{1-c_3/2/\gamma}}}{\sqrt{1 - c_3/2/\gamma}}. \end{aligned} \quad (133)$$

As in Section 2, to find an optimum in (132) we will compute the derivatives of $\zeta_{\alpha,\beta}^{(box)}(c_3, \nu, A_0)$ with respect to c_3 , ν , and A_0 and solve the following system of three equations

$$\frac{d\zeta_{\alpha,\beta}^{(box)}(c_3, \nu, A_0)}{dc_3} = \frac{d\zeta_{\alpha,\beta}^{(box)}(c_3, \nu, A_0)}{d\nu} = \frac{d\zeta_{\alpha,\beta}^{(box)}(c_3, \nu, A_0)}{dA_0} = 0. \quad (134)$$

We start by recognizing that the considerations related to the derivative with respect to c_3 are the same as in Section 2. Namely,

$$\frac{d\zeta_{\alpha,\beta}^{(box)}(c_3, \nu, A_0)}{dc_3} = -c_3 + \frac{c_3}{1 - A_0^2} + \frac{c_3 - \sqrt{(c_3)^2 + 4\alpha}}{2}, \quad (135)$$

and after setting further the above derivative to zero one has

$$c_3 = \frac{(1 - A_0^2)\sqrt{\alpha}}{A_0}. \quad (136)$$

The derivatives with respect to ν and A_0 are different and a bit more involved. For the derivative with respect to ν we have

$$\frac{d\zeta_{\alpha,\beta}^{(box)}(c_3, \nu, A_0)}{d\nu} = \frac{d}{d\nu} \left(-\frac{c_3^2}{2} + I_{sph} + \mu(1 - \beta) \log w_1 + (1 - \mu)(1 - \beta) \log w_2 + \beta \log w_3 + \frac{c_3^2}{2(1 - A_0^2)} \right)$$

$$\begin{aligned}
&= \frac{\mu(1-\beta)}{w_1} \left(\frac{(1-A_0^2)\nu}{A_0^2} e^{\frac{(1-A_0^2)\nu^2}{2A_0^2}} \operatorname{erfc}\left(\frac{\nu}{\sqrt{2}A_0}\right) - \frac{1-A_0^2}{A_0^2} \frac{2e^{-\frac{\nu^2}{2}}}{\sqrt{2}\sqrt{\pi}} \right) \\
&\quad - \frac{(1-\mu)(1-\beta)}{w_2} \left(-\frac{(1-A_0^2)\nu}{A_0^2} e^{\frac{(1-A_0^2)\nu^2}{2A_0^2}} \operatorname{erfc}\left(\frac{-\nu}{\sqrt{2}A_0}\right) - \frac{1-A_0^2}{A_0^2} \frac{2e^{-\frac{\nu^2}{2}}}{\sqrt{2}\sqrt{\pi}} \right) + \frac{1-A_0^2}{A_0^2} \beta \nu.
\end{aligned} \tag{137}$$

To facilitate the exposition we recall on the following from (25)

$$\begin{aligned}
y_1 &= \frac{\nu}{\sqrt{2}} \\
y_2 &= \frac{\nu}{\sqrt{2}A_0} = \frac{y_1}{A_0}.
\end{aligned} \tag{138}$$

Similarly to (139) we also set

$$\begin{aligned}
z_{1,\nu}^{(box)} &= \frac{\mu(1-\beta)y_1 \left(\sqrt{2}y_2 e^{y_2^2} \operatorname{erfc}(y_2) - \sqrt{\frac{2}{\pi}} \right)}{y_2 e^{y_2^2} \operatorname{erfc}(y_2) + y_1 e^{y_1^2} \operatorname{erfc}(-y_1)} \\
z_{2,\nu}^{(box)} &= -\frac{(1-\mu)(1-\beta)y_1 \left(-\sqrt{2}y_2 e^{y_2^2} \operatorname{erfc}(-y_2) - \sqrt{\frac{2}{\pi}} \right)}{y_2 e^{y_2^2} \operatorname{erfc}(-y_2) + y_1 e^{y_1^2} \operatorname{erfc}(y_1)}.
\end{aligned} \tag{139}$$

Combining (133), (137), (138), and (139) we obtain the following analogous version of (27)

$$\frac{d\zeta_{\alpha,\beta}^{(box)}(c_3, \nu, A_0)}{d\nu} = \frac{1-A_0^2}{A_0^2} \left(z_{1,\nu}^{(box)} + z_{2,\nu}^{(box)} + \beta\sqrt{2}y_1 \right). \tag{140}$$

Together, (139) and (140), then also give

$$\begin{aligned}
&\frac{d\zeta_{\alpha,\beta}^{(box)}(c_3, \nu, A_0)}{d\nu} = \frac{1-A_0^2}{A_0^2} \left(z_{1,\nu}^{(box)} + z_{2,\nu}^{(box)} + \beta\sqrt{2}y_1 \right) = 0 \\
\Rightarrow &\frac{\mu(1-\beta)y_1 \left(\sqrt{2}y_2 e^{y_2^2} \operatorname{erfc}(y_2) - \sqrt{\frac{2}{\pi}} \right)}{y_2 e^{y_2^2} \operatorname{erfc}(y_2) + y_1 e^{y_1^2} \operatorname{erfc}(-y_1)} - \frac{(1-\mu)(1-\beta)y_1 \left(-\sqrt{2}y_2 e^{y_2^2} \operatorname{erfc}(-y_2) - \sqrt{\frac{2}{\pi}} \right)}{y_2 e^{y_2^2} \operatorname{erfc}(-y_2) + y_1 e^{y_1^2} \operatorname{erfc}(y_1)} + \beta\sqrt{2}y_1 = 0.
\end{aligned} \tag{141}$$

To continue further transformation of the above derivative we will also rely on the following derivative with respect to A_0

$$\begin{aligned}
\frac{d\zeta_{\alpha,\beta}^{(box)}(c_3, \nu, A_0)}{dA_0} &= \frac{d}{dA_0} \left(-\frac{c_3^2}{2} + I_{sph} + \mu(1-\beta) \log w_1 + (1-\mu)(1-\beta) \log w_2 + \beta \log w_3 + \frac{c_3^2}{2(1-A_0^2)} \right) \\
&= \mu(1-\beta) \frac{d \log w_1}{dA_0} + (1-\mu)(1-\beta) \frac{d \log w_2}{dA_0} + \beta \frac{d \log w_3}{dA_0} + \frac{c_3^2 A_0}{(1-A_0^2)^2} \\
&= \mu(1-\beta) \frac{d \log w_1}{dA_0} + (1-\mu)(1-\beta) \frac{d \log w_2}{dA_0} + \beta \frac{-\nu^2 - A_0^2}{A_0^3} + \frac{\alpha}{A_0}.
\end{aligned} \tag{142}$$

In (30) and (31) we have already determined $\frac{d \log w_1}{dA_0}$ and $\frac{d \log w_2}{dA_0}$ as

$$\begin{aligned}\frac{d \log w_1}{dA_0} &= -\frac{e^{\frac{\nu^2}{2A_0^2}}(A_0^2 + \nu^2)\operatorname{erfc}(\frac{\nu}{\sqrt{2}A_0}) - \sqrt{\frac{2}{\pi}}A_0\nu}{A_0^3(e^{\frac{\nu^2}{2A_0^2}}\operatorname{erfc}(\frac{\nu}{\sqrt{2}A_0}) + A_0e^{\frac{\nu^2}{2}}(\operatorname{erf}(\frac{\nu}{\sqrt{2}}) + 1))} \\ \frac{d \log w_2}{dA_0} &= -\frac{e^{\frac{\nu^2}{2A_0^2}}(A_0^2 + \nu^2)\operatorname{erfc}(\frac{-\nu}{\sqrt{2}A_0}) + \sqrt{\frac{2}{\pi}}A_0\nu}{A_0^3(e^{\frac{\nu^2}{2A_0^2}}\operatorname{erfc}(\frac{-\nu}{\sqrt{2}A_0}) + A_0e^{\frac{\nu^2}{2}}(\operatorname{erf}(\frac{-\nu}{\sqrt{2}}) + 1))}\end{aligned}\quad (143)$$

Following closely (139) we set

$$\begin{aligned}z_{1,A_0}^{(box)} &= -\mu(1-\beta)\frac{y_2^2}{y_1}\frac{\left((1+2y_2^2)e^{y_2^2}\operatorname{erfc}(y_2) - \sqrt{2}y_2\sqrt{\frac{2}{\pi}}\right)}{(y_2e^{y_2^2}\operatorname{erfc}(y_2) + y_1e^{y_1^2}\operatorname{erfc}(-y_1))} \\ z_{2,A_0}^{(box)} &= -(1-\mu)(1-\beta)\frac{y_2^2}{y_1}\frac{\left((1+2y_2^2)e^{y_2^2}\operatorname{erfc}(-y_2) + \sqrt{2}y_2\sqrt{\frac{2}{\pi}}\right)}{(y_2e^{y_2^2}\operatorname{erfc}(-y_2) + y_1e^{y_1^2}\operatorname{erfc}(y_1))}.\end{aligned}\quad (144)$$

A combination of (137), (142), (143), and (144) gives

$$\begin{aligned}\frac{d\zeta_{\alpha,\beta}^{(box)}(c_3, \nu, A_0)}{dA_0} &= \mu(1-\beta)\frac{d \log w_1}{dA_0} + (1-\mu)(1-\beta)\frac{d \log w_2}{dA_0} + \beta\frac{-\nu^2 - A_0^2}{A_0^3} + \frac{\alpha}{A_0} \\ &= z_{1,A_0}^{(box)} + z_{2,A_0}^{(box)} + \frac{(\alpha - \beta - 2y_2^2\beta)y_2}{y_1}.\end{aligned}\quad (145)$$

Setting the derivative in (145) to zero and utilizing (144) we also have that

$$\frac{d\zeta_{\alpha,\beta}^{(box)}(c_3, \nu, A_0)}{dA_0} = z_{1,A_0}^{(box)} + z_{2,A_0}^{(box)} + \frac{(\alpha - \beta - 2y_2^2\beta)y_2}{y_1} = 0. \quad (146)$$

implies

$$\begin{aligned}-\mu(1-\beta)y_2\frac{\left((1+2y_2^2)e^{y_2^2}\operatorname{erfc}(y_2) - \sqrt{2}y_2\sqrt{\frac{2}{\pi}}\right)}{y_2e^{y_2^2}\operatorname{erfc}(y_2) + y_1e^{y_1^2}\operatorname{erfc}(-y_1)} - (1-\mu)(1-\beta)y_2\frac{\left((1+2y_2^2)e^{y_2^2}\operatorname{erfc}(-y_2) + \sqrt{2}y_2\sqrt{\frac{2}{\pi}}\right)}{y_2e^{y_2^2}\operatorname{erfc}(-y_2) + y_1e^{y_1^2}\operatorname{erfc}(y_1)} \\ + \alpha - \beta - 2y_2^2\beta = 0.\end{aligned}\quad (147)$$

Combining further (141) and (147) we obtain

$$\begin{aligned}\frac{d\zeta_{\alpha,\beta}^{(box)}(c_3, \nu, A_0)}{dA_0} &= z_{1,A_0}^{(box)} + z_{2,A_0}^{(box)} + \frac{(\alpha - \beta - 2y_2^2\beta)y_2}{y_1} = 0 \\ \implies &-\frac{\mu(1-\beta)y_2e^{y_2^2}\operatorname{erfc}(y_2)}{y_2e^{y_2^2}\operatorname{erfc}(y_2) + y_1e^{y_1^2}\operatorname{erfc}(-y_1)} - \frac{(1-\mu)(1-\beta)y_2e^{y_2^2}\operatorname{erfc}(-y_2)}{y_2e^{y_2^2}\operatorname{erfc}(-y_2) + y_1e^{y_1^2}\operatorname{erfc}(y_1)} + \alpha - \beta - 2\beta y_2^2 + 2\beta y_2^2 = 0 \\ \implies &-\frac{\mu(1-\beta)\sqrt{\frac{1}{\pi}}}{y_2e^{y_2^2}\operatorname{erfc}(y_2) + y_1e^{y_1^2}\operatorname{erfc}(-y_1)} + \frac{(1-\mu)(1-\beta)\sqrt{\frac{1}{\pi}}}{y_2e^{y_2^2}\operatorname{erfc}(-y_2) + y_1e^{y_1^2}\operatorname{erfc}(y_1)} + \alpha = 0 \\ \implies &-\frac{\mu(1-\beta)}{y_2e^{y_2^2}\operatorname{erfc}(y_2) + y_1e^{y_1^2}\operatorname{erfc}(-y_1)} + \frac{(1-\mu)(1-\beta)}{y_2e^{y_2^2}\operatorname{erfc}(-y_2) + y_1e^{y_1^2}\operatorname{erfc}(y_1)} + \alpha\sqrt{\pi} = 0.\end{aligned}\quad (148)$$

Additionally, a simple algebraic transformation of (141) gives

$$\begin{aligned} \frac{(1-\mu)(1-\beta)}{y_2 e^{y_2^2} \operatorname{erfc}(-y_2) + y_1 e^{y_1^2} \operatorname{erfc}(y_1)} &= \frac{\mu(1-\beta)}{(y_2 e^{y_2^2} \operatorname{erfc}(y_2) + y_1 e^{y_1^2} \operatorname{erfc}(-y_1))} \frac{\left(\sqrt{2} y_2 e^{y_2^2} \operatorname{erfc}(y_2) - \sqrt{\frac{2}{\pi}}\right)}{\left(-\sqrt{2} y_2 e^{y_2^2} \operatorname{erfc}(-y_2) - \sqrt{\frac{2}{\pi}}\right)} \\ &\quad + \frac{\beta\sqrt{2}}{\left(-\sqrt{2} y_2 e^{y_2^2} \operatorname{erfc}(-y_2) - \sqrt{\frac{2}{\pi}}\right)}. \end{aligned} \quad (149)$$

After plugging the right side of (149) in (148) we obtain

$$\begin{aligned} \frac{d\zeta_{\alpha,\beta}^{(box)}(c_3, \nu, A_0)}{dA_0} &= z_{1,A_0}^{(box)} + z_{2,A_0}^{(box)} + \frac{(\alpha - \beta - 2\beta y_2^2)y_2}{y_1} = 0 \\ \Rightarrow & -\frac{\mu(1-\beta)}{y_2 e^{y_2^2} \operatorname{erfc}(y_2) + y_1 e^{y_1^2} \operatorname{erfc}(-y_1)} + \frac{(1-\mu)(1-\beta)}{y_2 e^{y_2^2} \operatorname{erfc}(-y_2) + y_1 e^{y_1^2} \operatorname{erfc}(y_1)} + \alpha\sqrt{\pi} = 0 \\ \Rightarrow & \frac{\mu(1-\beta)}{y_2 e^{y_2^2} \operatorname{erfc}(y_2) + y_1 e^{y_1^2} \operatorname{erfc}(-y_1)} \left(-1 + \frac{\left(\sqrt{2} y_2 e^{y_2^2} \operatorname{erfc}(y_2) - \sqrt{\frac{2}{\pi}}\right)}{\left(-\sqrt{2} y_2 e^{y_2^2} \operatorname{erfc}(-y_2) - \sqrt{\frac{2}{\pi}}\right)} \right) \\ & + \frac{\beta\sqrt{2}}{\left(-\sqrt{2} y_2 e^{y_2^2} \operatorname{erfc}(-y_2) - \sqrt{\frac{2}{\pi}}\right)} + \alpha\sqrt{\pi} = 0 \\ \Rightarrow & y_2 e^{y_2^2} \operatorname{erfc}(y_2) + y_1 e^{y_1^2} \operatorname{erfc}(-y_1) = \mu(1-\beta) \left(\frac{y_2 e^{y_2^2} \operatorname{erfc}(-y_2) + y_2 e^{y_2^2} \operatorname{erfc}(y_2)}{\alpha(\sqrt{\pi} y_2 e^{y_2^2} \operatorname{erfc}(-y_2) + 1) - \beta} \right) \\ \Rightarrow & y_1 e^{y_1^2} \operatorname{erfc}(-y_1) = \left(\frac{\mu(1-\beta) 2y_2 e^{y_2^2}}{\alpha(\sqrt{\pi} y_2 e^{y_2^2} \operatorname{erfc}(-y_2) + 1) - \beta} \right) - y_2 e^{y_2^2} \operatorname{erfc}(y_2) \triangleq f_1^{(box)}(y_2; \alpha, \beta, \mu). \end{aligned} \quad (150)$$

Similarly to what was done in (38), one can also combine (148) and (149) in the following alternative way to obtain

$$\begin{aligned} \frac{d\zeta_{\alpha,\beta}^{(box)}(c_3, \nu, A_0)}{dA_0} &= z_{1,A_0}^{(box)} + z_{2,A_0}^{(box)} + \frac{(\alpha - \beta - 2\beta y_2^2)y_2}{y_1} = 0 \\ \Rightarrow & -\frac{\mu(1-\beta)}{y_2 e^{y_2^2} \operatorname{erfc}(y_2) + y_1 e^{y_1^2} \operatorname{erfc}(-y_1)} + \frac{(1-\mu)(1-\beta)}{y_2 e^{y_2^2} \operatorname{erfc}(-y_2) + y_1 e^{y_1^2} \operatorname{erfc}(y_1)} + \alpha\sqrt{\pi} = 0 \\ \Rightarrow & \frac{(1-\mu)(1-\beta)}{y_2 e^{y_2^2} \operatorname{erfc}(-y_2) + y_1 e^{y_1^2} \operatorname{erfc}(y_1)} \left(1 - \frac{\left(-\sqrt{2} y_2 e^{y_2^2} \operatorname{erfc}(-y_2) - \sqrt{\frac{2}{\pi}}\right)}{\left(\sqrt{2} y_2 e^{y_2^2} \operatorname{erfc}(y_2) - \sqrt{\frac{2}{\pi}}\right)} \right) \\ & + \frac{\beta\sqrt{2}}{\left(\sqrt{2} y_2 e^{y_2^2} \operatorname{erfc}(y_2) - \sqrt{\frac{2}{\pi}}\right)} + \alpha\sqrt{\pi} = 0 \\ \Rightarrow & y_2 e^{y_2^2} \operatorname{erfc}(-y_2) + y_1 e^{y_1^2} \operatorname{erfc}(y_1) = -(1-\mu)(1-\beta) \left(\frac{y_2 e^{y_2^2} \operatorname{erfc}(-y_2) + y_2 e^{y_2^2} \operatorname{erfc}(y_2)}{\alpha(\sqrt{\pi} y_2 e^{y_2^2} \operatorname{erfc}(y_2) - 1) - \beta} \right) \\ \Rightarrow & y_1 e^{y_1^2} \operatorname{erfc}(y_1) = \left(\frac{(1-\mu)(1-\beta) 2y_2 e^{y_2^2}}{\alpha(-\sqrt{\pi} y_2 e^{y_2^2} \operatorname{erfc}(y_2) + 1) - \beta} \right) - y_2 e^{y_2^2} \operatorname{erfc}(-y_2) = -f_1^{(box)}(-y_2; \alpha, \beta, 1-\mu). \end{aligned} \quad (151)$$

A simple combination of (150) and (151) gives

$$\frac{1 + \operatorname{erf}(y_1)}{1 - \operatorname{erf}(y_1)} = \frac{\operatorname{erfc}(-y_1)}{\operatorname{erfc}(y_1)} = \frac{f_1^{(box)}(y_2; \alpha, \beta, \mu)}{-f_1^{(box)}(-y_2; \alpha, \beta, 1 - \mu)}. \quad (152)$$

Similarly to (40), after solving over $\operatorname{erf}(y_1)$ from (152) we obtain

$$\begin{aligned} \operatorname{erf}(y_1) &= \frac{f_1^{(box)}(y_2; \alpha, \beta, \mu) + f_1^{(box)}(-y_2; \alpha, \beta, 1 - \mu)}{f_1^{(box)}(y_2; \alpha, \beta, \mu) - f_1^{(box)}(-y_2; \alpha, \beta, 1 - \mu)} \\ \iff y_1 &= \operatorname{erfinv} \left(\frac{f_1^{(box)}(y_2; \alpha, \beta, \mu) + f_1^{(box)}(-y_2; \alpha, \beta, 1 - \mu)}{f_1^{(box)}(y_2; \alpha, \beta, \mu) - f_1^{(box)}(-y_2; \alpha, \beta, 1 - \mu)} \right). \end{aligned} \quad (153)$$

Combining (150) (or (151)) and (153) we also have

$$2\operatorname{erfinv} \left(\frac{f_1^{(box)}(y_2; \alpha, \beta, \mu) + f_1^{(box)}(-y_2; \alpha, \beta, 1 - \mu)}{f_1^{(box)}(y_2; \alpha, \beta, \mu) - f_1^{(box)}(-y_2; \alpha, \beta, 1 - \mu)} \right) \frac{e^{\operatorname{erfinv} \left(\frac{f_1^{(box)}(y_2; \alpha, \beta, \mu) + f_1^{(box)}(-y_2; \alpha, \beta, 1 - \mu)}{f_1^{(box)}(y_2; \alpha, \beta, \mu) - f_1^{(box)}(-y_2; \alpha, \beta, 1 - \mu)} \right)^2}}{f_1^{(box)}(y_2; \alpha, \beta, \mu) - f_1^{(box)}(-y_2; \alpha, \beta, 1 - \mu)} = 1. \quad (154)$$

(154) can be used to determine y_2 which can be reused to obtain y_1 from (150) and (151) (basically (153)). Using (131), (136), and (138), one can then obtain ν , A_0 , c_3 , and γ as in (42)

$$\begin{aligned} \nu &= y_2 \sqrt{2} \\ A_0 &= \frac{y_1}{y_2} \\ c_3 &= \frac{(1 - A_0^2) \sqrt{\alpha}}{A_0} \\ \gamma &= \frac{c_3}{2(1 - A_0)}. \end{aligned} \quad (155)$$

Finally, after all of the above is determined one can compute $\zeta_{\alpha, \beta}^{(box)}(c_3, \nu, A_0)$ in (133) following the methodology showcased in (156) and (157). As in (156), we first note that from [17] one has

$$I_{sph} = -\frac{(1 - A_0^2)\alpha}{2} + \alpha \log(A_0). \quad (156)$$

A combination of (150), (151), (155), and (156) then finally produces

$$\begin{aligned} I_{err, u}^{(box, ub)} &= \zeta_{\alpha, \beta}^{(box)}(c_3, \nu, A_0) = -\frac{c_3^2}{2} + I_{sph} + \mu(1 - \beta) \log w_1 + (1 - \mu)(1 - \beta) \log w_2 + \beta \log w_3 + \frac{c_3^2}{2(1 - A_0^2)} \\ &= -\frac{c_3^2}{2} - \frac{(1 - A_0^2)\alpha}{2} + \alpha \log(A_0) + \mu(1 - \beta) \log w_1 + (1 - \mu)(1 - \beta) \log w_2 + \beta \log w_3 + \frac{c_3^2}{2(1 - A_0^2)} \\ &= \alpha \log(A_0) + \mu(1 - \beta) \log w_1 + (1 - \mu)(1 - \beta) \log w_2 + \beta \log w_3 \\ &= \alpha \log \left(\frac{y_1}{y_2} \right) + \mu(1 - \beta) \log \left(e^{y_2^2} y_2 \operatorname{erfc}(y_2) + e^{y_1^2} y_1 \operatorname{erfc}(-y_1) \right) \\ &\quad + (1 - \mu)(1 - \beta) \log \left(e^{y_2^2} y_2 \operatorname{erfc}(-y_2) + e^{y_1^2} y_1 \operatorname{erfc}(y_1) \right) - (1 - \beta) y_1^2 - (1 - \beta) \log(y_1) \\ &\quad - (1 - \beta) \log(2) + \beta(y_2^2 - y_1^2) - \beta \log \left(\frac{y_1}{y_2} \right) \\ &= \alpha \log \left(\frac{y_1}{y_2} \right) + \mu(1 - \beta) \log \left(\frac{\mu(1 - \beta) 2 y_2 e^{y_2^2}}{\alpha(\sqrt{\pi} y_2 e^{y_2^2} \operatorname{erfc}(-y_2) + 1) - \beta} \right) \end{aligned}$$

$$\begin{aligned}
& +(1-\mu)(1-\beta) \log \left(\frac{(1-\mu)(1-\beta)2y_2e^{y_2^2}}{\alpha(-\sqrt{\pi}y_2e^{y_2^2}\operatorname{erfc}(y_2)+1)-\beta} \right) \\
& -(1-\beta)y_1^2 - (1-\beta) \log(y_1) - (1-\beta) \log(2) + \beta(y_2^2 - y_1^2) - \beta \log \left(\frac{y_1}{y_2} \right) \\
= & (\alpha-1) \log \left(\frac{y_1}{y_2} \right) + \mu(1-\beta) \log \left(\frac{\mu(1-\beta)}{\alpha(\sqrt{\pi}y_2e^{y_2^2}\operatorname{erfc}(-y_2)+1)-\beta} \right) \\
& +(1-\mu)(1-\beta) \log \left(\frac{(1-\mu)(1-\beta)}{\alpha(-\sqrt{\pi}y_2e^{y_2^2}\operatorname{erfc}(y_2)+1)-\beta} \right) + y_2^2 - y_1^2.
\end{aligned} \tag{157}$$

It is rather clear from the presented discussion that the choice for ν , A_0 , and c_3 given in (154) and (155) ensures that (134) is satisfied. In fact, not only that, one can also argue that this choice is besides being a stationary point also a global optimum in (132). As mentioned after (44) in Section 2, we will not pursue these considerations further here. Instead, we will below present a different set of considerations which we will then connect to what we presented above. At that time it will become clear that not only is the choice for ν , A_0 , and c_3 given in (154) and (155) precisely the one that solves the optimization in (132) but also precisely the one that determines $I_{err}^{(box)}(\alpha, \beta)$. We summarize the above results in the following theorem.

Theorem 12. *Assume the setup of Theorem 11 and assume that a pair (α, β) is given. Also, assume that $\alpha > \alpha_w$ where α_w is obtained from the phase transition curve as the value for α that corresponds to the given β . Set*

$$f_1^{(box)}(y_2; \alpha, \beta, \mu) \triangleq \frac{2\mu(1-\beta)y_2e^{y_2^2}}{\alpha(\sqrt{\pi}y_2e^{y_2^2}\operatorname{erfc}(-y_2)+1)-\beta} - y_2e^{y_2^2}\operatorname{erfc}(y_2). \tag{158}$$

Also let y_2 and y_1 satisfy the following **fundamental characterizations of the box ℓ_1 's LDP**:

$$2\operatorname{erf} \operatorname{inv} \left(\frac{f_1^{(box)}(y_2; \alpha, \beta, \mu) + f_1^{(box)}(-y_2; \alpha, \beta, 1-\mu)}{f_1^{(box)}(y_2; \alpha, \beta, \mu) - f_1^{(box)}(-y_2; \alpha, \beta, 1-\mu)} \right) \frac{e^{\operatorname{erf} \operatorname{inv} \left(\frac{f_1^{(box)}(y_2; \alpha, \beta, \mu) + f_1^{(box)}(-y_2; \alpha, \beta, 1-\mu)}{f_1^{(box)}(y_2; \alpha, \beta, \mu) - f_1^{(box)}(-y_2; \alpha, \beta, 1-\mu)} \right)^2}}{f_1^{(box)}(y_2; \alpha, \beta, \mu) - f_1^{(box)}(-y_2; \alpha, \beta, 1-\mu)} = 1 \tag{159}$$

and

$$y_1 = \operatorname{erf} \operatorname{inv} \left(\frac{f_1^{(box)}(y_2; \alpha, \beta, \mu) + f_1^{(box)}(-y_2; \alpha, \beta, 1-\mu)}{f_1^{(box)}(y_2; \alpha, \beta, \mu) - f_1^{(box)}(-y_2; \alpha, \beta, 1-\mu)} \right). \tag{160}$$

Then

$$\begin{aligned}
I_{err}^{(box)}(\alpha, \beta) & \triangleq \lim_{n \rightarrow \infty} \frac{\log P_{err}^{(box)}}{n} \leq (\alpha-1) \log \left(\frac{y_1}{y_2} \right) + \mu(1-\beta) \log \left(\frac{\mu(1-\beta)}{\alpha(\sqrt{\pi}e^{y_2^2}y_2\operatorname{erfc}(-y_2)+1)-\beta} \right) \\
& \quad + (1-\mu)(1-\beta) \log \left(\frac{(1-\mu)(1-\beta)}{\alpha(-\sqrt{\pi}e^{y_2^2}y_2\operatorname{erfc}(y_2)+1)-\beta} \right) + y_2^2 - y_1^2 \\
& \triangleq I_{ldp}^{(box)}(\alpha, \beta).
\end{aligned} \tag{161}$$

Moreover, for the above choice of y_2 and y_1 , ν , A_0 , c_3 , and γ in (155) achieve an optimum in (132) (and ultimately in (129)).

Proof. Follows from the above discussion. \square

As in Section 2, the above upper tail results remain correct in the lower tail regime as well. A short

argument that we for the completeness present below (and that closely follows what we presented in Section 2.2.3) confirms this.

3.1.3 Lower tail

As in Section 2.2.3, we rely on the strategy introduced in [17] and write

$$P_{cor} \leq \min_{t_1} \min_{c_3 \geq 0} E e^{c_3 \|\mathbf{g}\|^2} E e^{-c_3 w(\mathbf{h}, S_w^{(box)})} e^{-c_3 t_1} / P(g \geq t_1), \quad (162)$$

where $P_{cor} = 1 - P_{err}^{(box)}$ is the probability that (118) does produce the solution of (2). Following (50) we also introduce the rate of P_{cor} 's decay

$$I_{cor}^{(box)}(\alpha, \beta) \triangleq \lim_{n \rightarrow \infty} \frac{\log P_{cor}}{n}. \quad (163)$$

The lower tail analogue to Theorem 11 is then the following theorem.

Theorem 13. *Assume the setup of Theorem 11. Then*

$$\begin{aligned} I_{cor}^{(box)}(\alpha, \beta) &\triangleq \lim_{n \rightarrow \infty} \frac{\log P_{cor}}{n} \\ &\leq \min_{c_3 \geq 0} \left(-\frac{c_3^2}{2} + I_{sph}^+ + \max_{\nu \geq 0, \gamma^{(s)} \geq 0} (\mu(1-\beta) \log w_1 + (1-\mu)(1-\beta) \log w_2 + \beta \log w_3 - c_3 \gamma) \right) \\ &\triangleq I_{cor,l}^{(box,ub)}(\alpha, \beta), \end{aligned} \quad (164)$$

where

$$\begin{aligned} I_{sph}^+ &= \widehat{\gamma}_+ c_3 - \frac{\alpha d}{2} \log \left(1 - \frac{c_3}{2\widehat{\gamma}_+} \right) \\ \widehat{\gamma}_+ &= \frac{2c_3 + \sqrt{4c_3^2 + 16\alpha}}{8} \\ w_1 &= \frac{1}{2} \left(\frac{1}{\sqrt{2\pi}} \int_h e^{-h^2/2} e^{-c_3 \max(h-\nu, 0)^2/4/\gamma} dh = \frac{e^{\frac{-c_3 \nu^2/4/\gamma}{1+c_3/2/\gamma}} \operatorname{erfc} \left(\frac{\nu}{\sqrt{2}\sqrt{1+c_3/2/\gamma}} \right) + \operatorname{erf} \left(\frac{\nu}{\sqrt{2}} \right) + 1}{\sqrt{1+c_3/2/\gamma}} \right) \\ w_2 &= \frac{1}{2} \left(\frac{1}{\sqrt{2\pi}} \int_h e^{-h^2/2} e^{-c_3 \max(h+\nu, 0)^2/4/\gamma} dh = \frac{e^{\frac{-c_3 \nu^2/4/\gamma}{1+c_3/2/\gamma}} \operatorname{erfc} \left(\frac{-\nu}{\sqrt{2}\sqrt{1+c_3/2/\gamma}} \right) + \operatorname{erf} \left(\frac{-\nu}{\sqrt{2}} \right) + 1}{\sqrt{1+c_3/2/\gamma}} \right) \\ w_3 &= \frac{1}{\sqrt{2\pi}} \int_h e^{-h^2/2} e^{-c_3(h+\nu)^2/4/\gamma} dh = \frac{e^{\frac{-c_3 \nu^2/4/\gamma}{1+c_3/2/\gamma}}}{\sqrt{1+c_3/2/\gamma}}. \end{aligned} \quad (165)$$

Proof. Follows in exactly the same way as the proof of Theorem 6 and ultimately the corresponding result for the lower tail of the standard ℓ_1 LDP in [17]. \square

Instead solving the above optimization problem, one can just quickly observe that the change $c_3 \rightarrow -c_3$ gives as in Section 3.1.2

$$A_0 \triangleq \sqrt{1 - \frac{c_3}{2\gamma}}, \quad (166)$$

and

$$I_{cor,l}^{(box,ub)}(\alpha, \beta) \triangleq \min_{c_3 \leq 0} \max_{\nu \geq 0, A_0 \leq 1} \zeta_{\alpha, \beta}^{(box)}(c_3, \nu, A_0) \quad (167)$$

where

$$\begin{aligned}
\zeta_{\alpha,\beta}^{(box)}(c_3, \nu, A_0) &= \left(-\frac{c_3^2}{2} + I_{sph}^+ + \mu(1-\beta) \log w_1 + (1-\mu)(1-\beta) \log w_2 + \beta \log w_3 + \frac{c_3^2}{2(1-A_0^2)} \right) \\
I_{sph}^+ &= -\widehat{\gamma}^+ c_3 - \frac{\alpha}{2} \log \left(1 + \frac{c_3}{2\widehat{\gamma}^+} \right) \\
\widehat{\gamma}^+ &= \frac{-c_3 + \sqrt{c_3^2 + 4\alpha}}{4} = -\widehat{\gamma},
\end{aligned} \tag{168}$$

and w_1 , w_2 , and w_3 are as in (133). One then observes that $\zeta_{\alpha,\beta}^{(box)}(c_3, \nu, A_0)$ defined in (168) is exactly the same as the corresponding one in (133) which means that one can proceed with the computation of all the derivatives as earlier and the values we have chosen for c_3 , ν , γ , and A_0 in the upper tail regime will have the same form. The following theorem summarizes the final results.

Theorem 14. *Assume the setup of Theorem 12 and assume that a pair (α, β) is given. Differently from Theorem 12, assume that $\alpha < \alpha_w$. Also let y_2 and y_1 satisfy the **fundamental box** ℓ_1 's **LDP characterizations** as in Theorem 12. Then choosing ν , c_3 , and γ in the optimization problem in (164) as ν , $-c_3$, and γ from Theorem 12 (or equivalently, choosing ν , c_3 , and A_0 in the optimization problem in (167) as ν , c_3 , and A_0 from Theorem 12) gives needed $\zeta_{\alpha,\beta}^{(box)}(c_3, \nu, A_0)$.*

Proof. Follows from the considerations leading to Theorem 12. \square

Similarly to what was observed after Theorem 7 we have for $\alpha < \alpha_w$, $y_1 > y_2$ which means $A_0 > 1$ and finally $c_3 < 0$. This is of course different from the upper tail regime (i.e. Theorem 12), where the reasoning is reversed and $c_3 > 0$.

3.2 High-dimensional geometry

In this section we provide an analysis that is analogous to the one provided in Section 2.3 for the binary ℓ_1 . As in Section 2.3, the analysis that we will present below relies on a high-dimensional integral geometry approach. Many aspects of the analysis presented in Section 2.3 will be directly applicable here as well. Some of them though will be different. As usual, we will focus on highlighting the key differences.

As earlier, we will here again mostly focus on the upper tail regime (the results for the lower tail will automatically follow). Mathematically, the upper tail regime will assume that we are given a pair (α, β) such that $\alpha > \alpha_w$, where α_w is as in Theorems 12, 13, and 14.

We will rely on the following observations from [15]

$$\Psi_{net}^{(box)}(\alpha, \beta) = I_{err}^{(box)}(\alpha, \beta) \triangleq \lim_{n \rightarrow \infty} \frac{\log P_{err}^{(box)}}{n} = \max_{\gamma_g \in (0, \min(1-\alpha, (1-\mu)(1-\beta)))} \left(\Psi_{com}^{(box)} + \Psi_{int}^{(box)} + \Psi_{ext}^{(box)} \right), \tag{169}$$

where

$$\begin{aligned}
H(x) &= x \log(x) + (1-x) \log(1-x) \\
\Psi_{com}^{(box)} &= -\mu(1-\beta) H\left(\frac{1-\alpha-\gamma_g}{\mu(1-\beta)}\right) - (1-\mu)(1-\beta) H\left(\frac{(1-\mu)(1-\beta)-\gamma_g}{(1-\mu)(1-\beta)}\right) \\
\Psi_{int}^{(box)} &= \min_{y_i \geq 0} (\alpha y_i^2 + ((1-\beta)\mu - (1-\alpha-\gamma_g)) \log(\operatorname{erfc}(y_i)) + ((1-\beta)(1-\mu) - \gamma_g) \log(\operatorname{erfc}(-y_i))) \\
&\quad - (\alpha - \beta) \log(2) \\
\Psi_{ext}^{(box)} &= \max_{y_e \geq 0} (-\alpha y_e^2 + (1-\alpha-\gamma_g) \log(\operatorname{erfc}(-y_e)) + \gamma_g \log(\operatorname{erfc}(y_e))) - (1-\alpha) \log(2).
\end{aligned} \tag{170}$$

As in Section 2, instead of solving the above problem numerically, we will here raise the bar a bit higher and

look for an explicit solution. We will start with the following analogue to (58)

$$\max_{\gamma_g \in (0, \beta), y_e \geq 0} \min_{y_i \geq 0} \zeta_{\alpha, \beta}^{(box)}(\gamma_g, y_e, y_i) \quad (171)$$

where

$$\begin{aligned} \zeta_{\alpha, \beta}^{(box)}(\gamma_g, y_e, y_i) &= -\mu(1-\beta)H\left(\frac{1-\alpha-\gamma_g}{\mu(1-\beta)}\right) - (1-\mu)(1-\beta)H\left(\frac{(1-\mu)(1-\beta)-\gamma_g}{(1-\mu)(1-\beta)}\right) \\ &\quad + \alpha y_i^2 + ((1-\beta)\mu - (1-\alpha-\gamma_g))\log(\operatorname{erfc}(y_i)) + ((1-\beta)(1-\mu) - \gamma_g)\log(\operatorname{erfc}(-y_i)) \\ &\quad - (\alpha - \beta)\log(2) \\ &\quad + (-\alpha y_e^2 + (1-\alpha-\gamma_g)\log(\operatorname{erfc}(-y_e)) + \gamma_g \log(\operatorname{erfc}(y_e))) - (1-\alpha)\log(2). \end{aligned} \quad (172)$$

Following further Section 2, we below briefly discuss a few useful properties of $\zeta_{\alpha, \beta}^{(box)}(\gamma_g, y_e, y_i)$.

3.2.1 Properties of $\zeta_{\alpha, \beta}^{(box)}(\gamma_g, y_e, y_i)$

Here we will quickly establish that for any fixed $(\alpha, \beta) \in (0, 1) \times (0, \alpha)$, $\zeta_{\alpha, \beta}^{(box)}(\gamma_g, y_e, y_i)$ is concave in γ_g and y_e and convex in y_i in the optimizing domain (these are precisely the same properties that $\zeta_{\alpha, \beta}(\gamma_g, y_e, y_i)$ exhibits).

Concavity in y_e follows automatically from the concavity of $\zeta_{\alpha, \beta}(\gamma_g, y_e, y_i)$. Concavity in γ_g follows in a very similar manner. We first compute the first derivative with respect to γ_g

$$\begin{aligned} \frac{d\zeta_{\alpha, \beta}^{(box)}(\gamma_g, y_e, y_i)}{d\gamma_g} &= -\mu(1-\beta)\frac{dH\left(\frac{1-\alpha-\gamma_g}{\mu(1-\beta)}\right)}{d\gamma_g} - (1-\mu)(1-\beta)\frac{dH\left(\frac{(1-\mu)(1-\beta)-\gamma_g}{(1-\mu)(1-\beta)}\right)}{d\gamma_g} \\ &\quad + \log\left(\frac{\operatorname{erfc}(y_i)\operatorname{erfc}(y_e)}{\operatorname{erfc}(-y_i)\operatorname{erfc}(-y_e)}\right) \\ &= \log\left(\frac{\frac{1-\alpha-\gamma_g}{\mu(1-\beta)}}{1-\frac{1-\alpha-\gamma_g}{\mu(1-\beta)}}\right) + \log\left(\frac{\frac{(1-\mu)(1-\beta)-\gamma_g}{(1-\mu)(1-\beta)}}{1-\frac{(1-\mu)(1-\beta)-\gamma_g}{(1-\mu)(1-\beta)}}\right) + \log\left(\frac{\operatorname{erfc}(y_i)\operatorname{erfc}(y_e)}{\operatorname{erfc}(-y_i)\operatorname{erfc}(-y_e)}\right) \\ &= \log\left(\frac{1-\alpha-\gamma_g}{\mu(1-\beta)-(1-\alpha-\gamma_g)}\right) + \log\left(\frac{(1-\mu)(1-\beta)-\gamma_g}{\gamma_g}\right) \\ &\quad + \log\left(\frac{\operatorname{erfc}(y_i)\operatorname{erfc}(y_e)}{\operatorname{erfc}(-y_i)\operatorname{erfc}(-y_e)}\right), \end{aligned} \quad (173)$$

and then the second one as well

$$\begin{aligned} \frac{d^2\zeta_{\alpha, \beta}^{(box)}(\gamma_g, y_e, y_i)}{d\gamma_g^2} &= \frac{d\left(\log\left(\frac{1-\alpha-\gamma_g}{\mu(1-\beta)-(1-\alpha-\gamma_g)}\right) + \log\left(\frac{(1-\mu)(1-\beta)-\gamma_g}{\gamma_g}\right) + \log\left(\frac{\operatorname{erfc}(y_i)\operatorname{erfc}(y_e)}{\operatorname{erfc}(-y_i)\operatorname{erfc}(-y_e)}\right)\right)}{d\gamma_g} \\ &= \frac{-\frac{1-\alpha-\gamma_g}{(\mu(1-\beta)-(1-\alpha-\gamma_g))^2} - \frac{1}{\mu(1-\beta)-(1-\alpha-\gamma_g)}}{\frac{1-\alpha-\gamma_g}{\mu(1-\beta)-(1-\alpha-\gamma_g)}} + \frac{-\frac{(1-\mu)(1-\beta)-\gamma_g}{\gamma_g^2} - \frac{1}{\gamma_g}}{\frac{(1-\mu)(1-\beta)-\gamma_g}{\gamma_g}} \\ &= \frac{-\mu(1-\beta)}{(1-\alpha-\gamma_g)(\mu(1-\beta)-(1-\alpha-\gamma_g))} + \frac{-(1-\mu)(1-\beta)}{\gamma_g((1-\mu)(1-\beta)-\gamma_g)} \\ &< 0. \end{aligned} \quad (174)$$

To check convexity in y_i we will need a bit of adaptation of the arguments from Section 2. We first recall that, as in Section 2, we will below rely on the following

$$\frac{d\log(\operatorname{erfc}(x))}{dx} = -\frac{2e^{-x^2}}{\sqrt{\pi}\operatorname{erfc}(x)} \quad \text{and} \quad \frac{d\log(\operatorname{erfc}(-x))}{dx} = \frac{2e^{-x^2}}{\sqrt{\pi}\operatorname{erfc}(-x)}, \quad (175)$$

and

$$\frac{d^2 \log(\operatorname{erfc}(x))}{dx^2} = \frac{4e^{-x^2} \sqrt{\pi} x \operatorname{erfc}(x) - 4e^{-2x^2}}{\pi \operatorname{erfc}(x)^2} \quad \text{and} \quad \frac{d^2 \log(\operatorname{erfc}(-x))}{dx^2} = \frac{-4e^{-x^2} \sqrt{\pi} x \operatorname{erfc}(-x) - 4e^{-2x^2}}{\pi \operatorname{erfc}(-x)^2}. \quad (176)$$

Now, we have for the first derivative with respect to y_1

$$\begin{aligned} \frac{d\zeta_{\alpha,\beta}^{(box)}(\gamma_g, y_e, y_i)}{dy_i} &= \frac{d(\alpha y_i^2 + ((1-\beta)\mu - (1-\alpha - \gamma_g)) \log(\operatorname{erfc}(y_i)) + ((1-\beta)(1-\mu) - \gamma_g) \log(\operatorname{erfc}(-y_i)))}{dy_i} \\ &= 2\alpha y_i - \frac{2((1-\beta)\mu - (1-\alpha - \gamma_g))e^{-y_i^2}}{\sqrt{\pi} \operatorname{erfc}(y_i)} + \frac{2((1-\beta)(1-\mu) - \gamma_g)e^{-y_i^2}}{\sqrt{\pi} \operatorname{erfc}(-y_i)}, \end{aligned} \quad (177)$$

and for the second

$$\begin{aligned} \frac{d^2 \zeta_{\alpha,\beta}^{(box)}(\gamma_g, y_e, y_i)}{dy_i^2} &= \frac{d^2(\alpha y_i^2 + ((1-\beta)\mu - (1-\alpha - \gamma_g)) \log(\operatorname{erfc}(y_i)) + ((1-\beta)(1-\mu) - \gamma_g) \log(\operatorname{erfc}(-y_i)))}{dy_i^2} \\ &= 2\alpha + ((1-\beta)\mu - (1-\alpha - \gamma_g)) \frac{4e^{-y_i^2} \sqrt{\pi} y_i \operatorname{erfc}(y_i) - 4e^{-2y_i^2}}{\pi \operatorname{erfc}(y_i)^2} \\ &\quad + ((1-\beta)(1-\mu) - \gamma_g) \frac{-4e^{-y_i^2} \sqrt{\pi} y_i \operatorname{erfc}(-y_i) - 4e^{-2y_i^2}}{\pi \operatorname{erfc}(-y_i)^2} \\ &\geq 2(\alpha - \beta) + ((1-\beta)\mu - (1-\alpha - \gamma_g)) \frac{4e^{-y_i^2} \sqrt{\pi} y_i \operatorname{erfc}(y_i) - 4e^{-2y_i^2}}{\pi \operatorname{erfc}(y_i)^2} \\ &\quad + ((1-\beta)(1-\mu) - \gamma_g) \frac{-4e^{-y_i^2} \sqrt{\pi} y_i \operatorname{erfc}(-y_i) - 4e^{-2y_i^2}}{\pi \operatorname{erfc}(-y_i)^2} \\ &= 2((1-\beta)\mu - (1-\alpha - \gamma_g)) \frac{\pi \operatorname{erfc}(y_i)^2 + 2e^{-y_i^2} \sqrt{\pi} y_i \operatorname{erfc}(y_i) - 2e^{-2y_i^2}}{\pi \operatorname{erfc}(y_i)^2} \\ &\quad + 2((1-\beta)(1-\mu) - \gamma_g) \frac{\pi \operatorname{erfc}(-y_i)^2 - 2e^{-y_i^2} \sqrt{\pi} y_i \operatorname{erfc}(-y_i) - 2e^{-2y_i^2}}{\pi \operatorname{erfc}(-y_i)^2} \\ &> 0, \end{aligned} \quad (178)$$

where the last inequality follows by repeating step by step the line of arguments after (65).

3.2.2 Solving the derivative equations

After establishing the above properties of $\zeta_{\alpha,\beta}^{(box)}(\gamma_g, y_e, y_i)$ we now focus on solving the following system of derivative equations

$$\frac{d\zeta_{\alpha,\beta}^{(box)}(\gamma_g, y_e, y_i)}{d\gamma_g} = \frac{d\zeta_{\alpha,\beta}^{(box)}(\gamma_g, y_e, y_i)}{dy_e} = \frac{d\zeta_{\alpha,\beta}^{(box)}(\gamma_g, y_e, y_i)}{dy_i} = 0. \quad (179)$$

The derivatives with respect to γ_g and y_i are computed in (173) and (177), respectively. We also recall on the derivative with respect to y_e from (67)

$$\begin{aligned} \frac{d\zeta_{\alpha,\beta}^{(box)}(\gamma_g, y_e, y_i)}{dy_e} &= \frac{d(-\alpha y_e^2 + (1-\alpha - \gamma_g) \log(\operatorname{erfc}(-y_e)) + \gamma_g \log(\operatorname{erfc}(y_e)))}{dy_e} \\ &= -2\alpha y_e + \frac{2(1-\alpha - \gamma_g)e^{-y_e^2}}{\sqrt{\pi} \operatorname{erfc}(-y_e)} - \frac{2\gamma_g e^{-y_e^2}}{\sqrt{\pi} \operatorname{erfc}(y_e)}. \end{aligned} \quad (180)$$

From (177) (after setting the derivative to zero) we have

$$\begin{aligned} \gamma_g \left(\frac{1}{\operatorname{erfc}(y_i)} + \frac{1}{\operatorname{erfc}(-y_i)} \right) &= \sqrt{\pi} \alpha e^{y_i^2} y_i - \frac{(1-\beta)\mu - (1-\alpha)}{\operatorname{erfc}(y_i)} + \frac{(1-\mu)(1-\beta)}{\operatorname{erfc}(-y_i)} \\ \iff \gamma_g &= \frac{\alpha}{2} \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) \operatorname{erfc}(-y_i) - \operatorname{erfc}(-y_i) \right) - (1-\beta)\mu + 1 - \frac{\operatorname{erfc}(y_i)\beta}{2}. \end{aligned} \quad (181)$$

Analogously to (77) we set

$$A_{box} = \frac{(1-\beta)\mu - (1-\alpha - \gamma_g)}{1 - \alpha - \gamma_g} \frac{\gamma_g}{(1-\mu)(1-\beta) - \gamma_g}. \quad (182)$$

Then (173) gives

$$\begin{aligned} A_{box} &= \frac{\operatorname{erfc}(y_i) \operatorname{erfc}(y_e)}{\operatorname{erfc}(-y_i) \operatorname{erfc}(-y_e)} \\ \iff \frac{\operatorname{erfc}(-y_e)}{\operatorname{erfc}(y_e)} &= \frac{\operatorname{erfc}(y_i)}{\operatorname{erfc}(-y_i) A_{box}} \\ \iff y_e &= \operatorname{erfinv} \left(\frac{\operatorname{erfc}(y_i) - A_{box} \operatorname{erfc}(-y_i)}{\operatorname{erfc}(y_i) + A_{box} \operatorname{erfc}(-y_i)} \right). \end{aligned} \quad (183)$$

As in Section 2, one can now use γ_g from (181) and y_e from (183) and combine it in the right side of (180). One can then equal the right side of (180) to zero and effectively obtain one equation with y_i as the only unknown. After determining y_i from such an equation one can use it to compute γ_g through (181) and y_e through (183). As in Section 2, we will focus on presenting the solution in a bit more explicit way and if possible in a way that is a bit more connected to what was presented in Section 3.1. To that end, we start, as usual, by setting the derivative in (180) to zero to obtain

$$\begin{aligned} \gamma_g \left(\frac{1}{\operatorname{erfc}(y_e)} + \frac{1}{\operatorname{erfc}(-y_e)} \right) &= -\sqrt{\pi} \alpha e^{y_e^2} y_e + \frac{1-\alpha}{\operatorname{erfc}(-y_e)} \\ \iff \gamma_g &= \frac{\alpha}{2} \left(-\sqrt{\pi} e^{y_e^2} y_e \operatorname{erfc}(-y_e) - 1 \right) \operatorname{erfc}(y_e) + \frac{\operatorname{erfc}(y_e)}{2}. \end{aligned} \quad (184)$$

Using γ_g from (181) in (182) gives

$$\begin{aligned} A_{box} &= \frac{\alpha + \frac{\alpha}{2} \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) \operatorname{erfc}(-y_i) - \operatorname{erfc}(-y_i) \right) - \frac{\operatorname{erfc}(y_i)\beta}{2}}{1 - \alpha - \left(\frac{\alpha}{2} \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) \operatorname{erfc}(-y_i) - \operatorname{erfc}(-y_i) \right) - (1-\beta)\mu + 1 - \frac{\operatorname{erfc}(y_i)\beta}{2} \right)} \\ &\times \frac{\frac{\alpha}{2} \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) \operatorname{erfc}(-y_i) - \operatorname{erfc}(-y_i) \right) - (1-\beta)\mu + 1 - \frac{\operatorname{erfc}(y_i)\beta}{2}}{-\frac{\alpha}{2} \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) \operatorname{erfc}(-y_i) - \operatorname{erfc}(-y_i) \right) - \frac{\operatorname{erfc}(-y_i)\beta}{2}}. \end{aligned} \quad (185)$$

Transforming a bit more A_{box} becomes

$$A_{box} = \frac{2(1-\beta)(1-\mu) + \left(\alpha \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) - 1 \right) + \beta \right) \operatorname{erfc}(-y_i)}{2(1-\beta)\mu - \left(\alpha \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(-y_i) + 1 \right) - \beta \right) \operatorname{erfc}(y_i)} \frac{\alpha \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(-y_i) + 1 \right) - \beta \operatorname{erfc}(y_i)}{\alpha \left(-\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) + 1 \right) - \beta \operatorname{erfc}(-y_i)}. \quad (186)$$

Combining (183) and (186) we also have

$$\frac{\operatorname{erfc}(-y_e)}{\operatorname{erfc}(y_e)} = \frac{\operatorname{erfc}(y_i)}{\operatorname{erfc}(-y_i) A_{box}} = \frac{2(1-\beta)\mu - \left(\alpha \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(-y_i) + 1 \right) - \beta \right) \operatorname{erfc}(y_i)}{2(1-\beta)(1-\mu) + \left(\alpha \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) - 1 \right) + \beta \right) \operatorname{erfc}(-y_i)}$$

$$\frac{\alpha \left(-\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) + 1 \right) - \beta}{\alpha \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(-y_i) + 1 \right) - \beta}. \quad (187)$$

We also recall on (150) and (151) and note that

$$\frac{\left(\frac{\mu(1-\beta)2y_2e^{y_2^2}}{\alpha(\sqrt{\pi}y_2e^{y_2^2}\operatorname{erfc}(-y_2)+1)-\beta} \right) - y_2e^{y_2^2}\operatorname{erfc}(y_2)}{\left(\frac{(1-\mu)(1-\beta)2y_2e^{y_2^2}}{\alpha(-\sqrt{\pi}y_2e^{y_2^2}\operatorname{erfc}(y_2)+1)-\beta} \right) - y_2e^{y_2^2}\operatorname{erfc}(-y_2)} = \frac{f_1^{(box)}(y_2; \alpha, \beta, \mu)}{-f_1^{(box)}(-y_2; \alpha, \beta, 1-\mu)}, \quad (188)$$

and

$$\begin{aligned} \frac{\mu(1-\beta)2y_2e^{y_2^2} - \left(\alpha(\sqrt{\pi}y_2e^{y_2^2}\operatorname{erfc}(-y_2)+1) - \beta \right) y_2e^{y_2^2}\operatorname{erfc}(y_2)}{(1-\mu)(1-\beta)2y_2e^{y_2^2} - \left(\alpha(-\sqrt{\pi}y_2e^{y_2^2}\operatorname{erfc}(y_2)+1) - \beta \right) y_2e^{y_2^2}\operatorname{erfc}(-y_2)} &\times \frac{\alpha \left(-\sqrt{\pi} e^{y_2^2} y_2 \operatorname{erfc}(y_2) + 1 \right) - \beta}{\alpha \left(\sqrt{\pi} e^{y_2^2} y_2 \operatorname{erfc}(-y_2) + 1 \right) - \beta} \\ &= \frac{f_1^{(box)}(y_2; \alpha, \beta, \mu)}{-f_1^{(box)}(-y_2; \alpha, \beta, 1-\mu)}. \end{aligned} \quad (189)$$

From (187) and (189) one also has

$$\begin{aligned} \frac{\operatorname{erfc}(-y_e)}{\operatorname{erfc}(y_e)} &= \frac{2(1-\beta)\mu - \left(\alpha \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(-y_i) + 1 \right) - \beta \right) \operatorname{erfc}(y_i)}{2(1-\beta)(1-\mu) + \left(\alpha \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) - 1 \right) + \beta \right) \operatorname{erfc}(-y_i)} \frac{\alpha \left(-\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) + 1 \right) - \beta}{\alpha \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(-y_i) + 1 \right) - \beta} \\ &= \frac{f_1^{(box)}(y_i; \alpha, \beta, \mu)}{-f_1^{(box)}(-y_i; \alpha, \beta, 1-\mu)}. \end{aligned} \quad (190)$$

A combination of (152), (153), and (190) gives

$$\begin{aligned} \operatorname{erf}(y_e) &= \frac{f_1^{(box)}(y_i; \alpha, \beta, \mu) + f_1^{(box)}(-y_i; \alpha, \beta, 1-\mu)}{f_1^{(box)}(y_i; \alpha, \beta, \mu) - f_1^{(box)}(-y_i; \alpha, \beta, 1-\mu)} \\ \iff y_e &= \operatorname{erfinv} \left(\frac{f_1^{(box)}(y_i; \alpha, \beta, \mu) + f_1^{(box)}(-y_i; \alpha, \beta, 1-\mu)}{f_1^{(box)}(y_i; \alpha, \beta, \mu) - f_1^{(box)}(-y_i; \alpha, \beta, 1-\mu)} \right). \end{aligned} \quad (191)$$

Connecting (181), (184), and ultimately (150) gives

$$\begin{aligned} \gamma_g &= \frac{\alpha}{2} \left(-\sqrt{\pi} e^{y_e^2} y_e \operatorname{erfc}(-y_e) - 1 \right) \operatorname{erfc}(y_e) + \frac{\operatorname{erfc}(y_e)}{2} \\ &= \frac{\alpha}{2} \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) \operatorname{erfc}(-y_i) - \operatorname{erfc}(-y_i) \right) - (1-\beta)\mu + 1 - \frac{\operatorname{erfc}(y_i)\beta}{2} \\ &= \frac{-f_1^{(box)}(-y_i; \alpha, \beta, 1-\mu) \left(\alpha \left(-\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) + 1 \right) - \beta \right)}{2e^{y_i^2} y_i}. \end{aligned} \quad (192)$$

Combining further (191) and (192) one obtains

$$\begin{aligned} \alpha \left(-\sqrt{\pi} e^{y_e^2} y_e \operatorname{erfc}(-y_e) - 1 \right) + 1 &= \frac{\left(f_1^{(box)}(y_i; \alpha, \beta, \mu) - f_1^{(box)}(-y_i; \alpha, \beta, 1-\mu) \right)}{2e^{y_i^2} y_i} \\ &\times \left(\alpha \left(-\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) + 1 \right) - \beta \right), \end{aligned} \quad (193)$$

and

$$e^{y_e^2} y_e \operatorname{erfc}(-y_e) = \frac{\left(f_1^{(box)}(y_i; \alpha, \beta, \mu) - f_1^{(box)}(-y_i; \alpha, \beta, 1 - \mu) \right) \left(\alpha \left(-\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) + 1 \right) - \beta \right)}{-\sqrt{\pi} 2 e^{y_i^2} y_i \alpha} + \frac{-2 e^{y_i^2} y_i + 2 e^{y_i^2} y_i \alpha}{-\sqrt{\pi} 2 e^{y_i^2} y_i \alpha}. \quad (194)$$

Similarly to what we done in Section 2, we will now argue that the right side in (194) is equal to $f_1^{(box)}(y_i; \alpha, \beta, \mu)$. That will follow if

$$\begin{aligned} & \left(f_1^{(box)}(y_i; \alpha, \beta, \mu) - f_1^{(box)}(-y_i; \alpha, \beta, 1 - \mu) \right) \left(\alpha \left(-\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) + 1 \right) - \beta \right) - 2 e^{y_i^2} y_i + 2 e^{y_i^2} y_i \alpha \\ &= -\sqrt{\pi} 2 e^{y_i^2} y_i \alpha f_1^{(box)}(y_i; \alpha, \beta, \mu), \end{aligned} \quad (195)$$

or

$$\begin{aligned} & -f_1^{(box)}(-y_i; \alpha, \beta, 1 - \mu) \left(\alpha \left(-\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) + 1 \right) - \beta \right) - 2 e^{y_i^2} y_i + 2 e^{y_i^2} y_i \alpha \\ &= f_1^{(box)}(y_i; \alpha, \beta, \mu) \left(\alpha \left(-\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(-y_i) - 1 \right) + \beta \right). \end{aligned} \quad (196)$$

Recalling again on (150) and (151), (196) can be rewritten in the following way

$$\begin{aligned} & 2(1 - \mu)(1 - \beta) - \left(\alpha \left(-\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) + 1 \right) - \beta \right) \operatorname{erfc}(-y_i) - 2 + 2\alpha \\ &= - \left(2\mu(1 - \beta) - \left(\alpha \left(\sqrt{\pi} y_i e^{y_i^2} \operatorname{erfc}(-y_i) + 1 \right) - \beta \right) \operatorname{erfc}(y_i) \right). \end{aligned} \quad (197)$$

Similarly to (92), (197) indeed holds and one then has that (196) holds as well. Through (195) we have then that the right side of (194) is indeed equal to $f_1^{(box)}(y_i; \alpha, \beta, \mu)$. One additional combination of (191) and (194) brings us finally to

$$\begin{aligned} & e^{y_e^2} y_e \operatorname{erfc}(-y_e) = f_1^{(box)}(y_i; \alpha, \beta, \mu) \\ \iff & 2 \operatorname{erfinv} \left(\frac{f_1^{(box)}(y_i; \alpha, \beta, \mu) + f_1^{(box)}(-y_i; \alpha, \beta, 1 - \mu)}{f_1^{(box)}(y_i; \alpha, \beta, \mu) - f_1^{(box)}(-y_i; \alpha, \beta, 1 - \mu)} \right) \frac{e^{\operatorname{erfinv} \left(\frac{f_1^{(box)}(y_i; \alpha, \beta, \mu) + f_1^{(box)}(-y_i; \alpha, \beta, 1 - \mu)}{f_1^{(box)}(y_i; \alpha, \beta, \mu) - f_1^{(box)}(-y_i; \alpha, \beta, 1 - \mu)} \right)^2}}}{f_1^{(box)}(y_i; \alpha, \beta, \mu) - f_1^{(box)}(-y_i; \alpha, \beta, 1 - \mu)} = 1. \end{aligned} \quad (198)$$

(198) is sufficient to compute y_i . One can then utilize such y_i and from (191) obtain y_e and from (181) or (184) γ_g (of course, a quick comparison of (198) and (191) on the one side and (154) and (153) on the other side gives $y_i = y_2$ and $y_e = y_1$). After these values for y_i , y_e , and γ_g are determined one can then use them to determine the optimal value of $\zeta_{\alpha, \beta}^{(box)}(\gamma_g, y_e, y_i)$ in (171) through (172). That eventually gives $I_{err}^{(box)}(\alpha, \beta)$ in (169). Following Section 2 though, we will below try to provide a bit more explicit connection between the optimal $\zeta_{\alpha, \beta}^{(box)}(\gamma_g, y_e, y_i)$ and what we presented in Section 2.2.

3.2.3 Computing $I_{err}^{(box)}(\alpha, \beta)$

As in Section 3.2.3 when we discussed $\zeta_{\alpha, \beta}^{(box)}(\gamma_g, y_e, y_i)$, we here recognize that instead of working directly with $\zeta_{\alpha, \beta}^{(box)}(\gamma_g, y_e, y_i)$ it will be a bit easier to first deal separately with each of $\Psi_{com}^{(box)}$, $\Psi_{int}^{(box)}$, and $\Psi_{ext}^{(box)}$. From this point on we assume that y_i , y_e , and γ_g take the values determined through the procedure explained

above. We start with $\Psi_{com}^{(box)}$ and write

$$\begin{aligned}
\Psi_{com}^{(box)} &= -\mu(1-\beta)H\left(\frac{1-\alpha-\gamma_g}{\mu(1-\beta)}\right) - (1-\mu)(1-\beta)H\left(\frac{(1-\mu)(1-\beta)-\gamma_g}{(1-\mu)(1-\beta)}\right) \\
&= -(1-\alpha-\gamma_g)\log\left(\frac{1-\alpha-\gamma_g}{\mu(1-\beta)}\right) - (\mu(1-\beta) - (1-\alpha-\gamma_g))\log\left(\frac{(\mu(1-\beta) - (1-\alpha-\gamma_g))}{\mu(1-\beta)}\right) \\
&\quad - ((1-\mu)(1-\beta) - \gamma_g)\log\left(\frac{(1-\mu)(1-\beta)-\gamma_g}{(1-\mu)(1-\beta)}\right) - \gamma_g\log\left(\frac{\gamma_g}{(1-\mu)(1-\beta)}\right) \\
&= -\frac{1}{2}\left(2\mu(1-\beta) - \left(\alpha\left(\sqrt{\pi}e^{y_i^2}y_i\text{erfc}(-y_i) + 1\right) - \beta\right)\text{erfc}(y_i)\right) \\
&\quad \times \log\left(\frac{\left(2\mu(1-\beta) - \left(\alpha\left(\sqrt{\pi}e^{y_i^2}y_i\text{erfc}(-y_i) + 1\right) - \beta\right)\text{erfc}(y_i)\right)}{2\mu(1-\beta)}\right) \\
&\quad - \frac{1}{2}\left(\left(\alpha\left(\sqrt{\pi}e^{y_i^2}y_i\text{erfc}(-y_i) + 1\right) - \beta\right)\text{erfc}(y_i)\right) \\
&\quad \times \log\left(\frac{\left(\alpha\left(\sqrt{\pi}e^{y_i^2}y_i\text{erfc}(-y_i) + 1\right) - \beta\right)\text{erfc}(y_i)}{2\mu(1-\beta)}\right) \\
&\quad - \frac{1}{2}\left(\left(\alpha\left(-\sqrt{\pi}e^{y_i^2}y_i\text{erfc}(y_i) + 1\right) - \beta\right)\text{erfc}(-y_i)\right) \\
&\quad \times \log\left(\frac{\left(\alpha\left(-\sqrt{\pi}e^{y_i^2}y_i\text{erfc}(y_i) + 1\right) - \beta\right)\text{erfc}(-y_i)}{2(1-\mu)(1-\beta)}\right) \\
&\quad - \frac{1}{2}\left(2(1-\mu)(1-\beta) - \left(\alpha\left(-\sqrt{\pi}e^{y_i^2}y_i\text{erfc}(y_i) + 1\right) - \beta\right)\text{erfc}(-y_i)\right) \\
&\quad \times \log\left(\frac{\left(2(1-\mu)(1-\beta) - \left(\alpha\left(-\sqrt{\pi}e^{y_i^2}y_i\text{erfc}(y_i) + 1\right) - \beta\right)\text{erfc}(-y_i)\right)}{2(1-\mu)(1-\beta)}\right).
\end{aligned} \tag{199}$$

For $\Psi_{int}^{(box)}$ we in a similar fashion have

$$\begin{aligned}
\Psi_{int}^{(box)} &= \alpha y_i^2 + ((1-\beta)\mu - (1-\alpha-\gamma_g))\log(\text{erfc}(y_i)) + ((1-\beta)(1-\mu) - \gamma_g)\log(\text{erfc}(-y_i)) \\
&\quad - (\alpha - \beta)\log(2) \\
&= \alpha y_i^2 + \frac{1}{2}\left(\left(\alpha\left(\sqrt{\pi}e^{y_i^2}y_i\text{erfc}(-y_i) + 1\right) - \beta\right)\text{erfc}(y_i)\right)\log(\text{erfc}(y_i)) \\
&\quad + \frac{1}{2}\left(\left(\alpha\left(-\sqrt{\pi}e^{y_i^2}y_i\text{erfc}(y_i) + 1\right) - \beta\right)\text{erfc}(-y_i)\right)\log(\text{erfc}(-y_i)) - (\alpha - \beta)\log(2).
\end{aligned} \tag{200}$$

Finally for $\Psi_{ext}^{(box)}$ we obtain

$$\begin{aligned}
\Psi_{ext}^{(box)} &= -\alpha y_e^2 + (1-\alpha-\gamma_g)\log(\text{erfc}(-y_e)) + \gamma_g\log(\text{erfc}(y_e)) - (1-\alpha)\log(2) \\
&= -\alpha y_e^2 + \frac{1}{2}\left(2\mu(1-\beta) - \left(\alpha\left(\sqrt{\pi}e^{y_i^2}y_i\text{erfc}(-y_i) + 1\right) - \beta\right)\text{erfc}(y_i)\right)\log(\text{erfc}(-y_e)) \\
&\quad + \frac{1}{2}\left(2(1-\mu)(1-\beta) + \left(\alpha\left(-\sqrt{\pi}e^{y_i^2}y_i\text{erfc}(y_i) + 1\right) - \beta\right)\text{erfc}(-y_i)\right)\log(\text{erfc}(y_e)) - (1-\alpha)\log(2).
\end{aligned} \tag{201}$$

Combining (199), (200), and (201) we also have

$$\begin{aligned}
\Psi_{net}^{(box)} = & -\frac{1}{2} \left(2\mu(1-\beta) - \left(\alpha \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(-y_i) + 1 \right) - \beta \right) \operatorname{erfc}(y_i) \right) \\
& \times \log \left(\frac{\left(2\mu(1-\beta) - \left(\alpha \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(-y_i) + 1 \right) - \beta \right) \operatorname{erfc}(y_i) \right)}{2\mu(1-\beta) \operatorname{erfc}(-y_e)} \right) \\
& -\frac{1}{2} \left(\left(\alpha \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(-y_i) + 1 \right) - \beta \right) \operatorname{erfc}(y_i) \right) \\
& \times \log \left(\frac{\left(\alpha \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(-y_i) + 1 \right) - \beta \right) \operatorname{erfc}(y_i)}{2\mu(1-\beta) \operatorname{erfc}(y_i)} \right) \\
& -\frac{1}{2} \left(\left(\alpha \left(-\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) + 1 \right) - \beta \right) \operatorname{erfc}(-y_i) \right) \\
& \times \log \left(\frac{\left(\alpha \left(-\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) + 1 \right) - \beta \right) \operatorname{erfc}(-y_i)}{2(1-\mu)(1-\beta) \operatorname{erfc}(-y_i)} \right) \\
& -\frac{1}{2} \left(2(1-\mu)(1-\beta) - \left(\alpha \left(-\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) + 1 \right) - \beta \right) \operatorname{erfc}(-y_i) \right) \\
& \times \log \left(\frac{\left(2(1-\mu)(1-\beta) - \left(\alpha \left(-\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) + 1 \right) - \beta \right) \operatorname{erfc}(-y_i) \right)}{2(1-\mu)(1-\beta) \operatorname{erfc}(y_e)} \right) \\
& + \alpha y_i^2 - \alpha y_e^2 - (1-\beta) \log(2).
\end{aligned} \tag{202}$$

Recalling once again (150) and (151), (202) can be further transformed

$$\begin{aligned}
\Psi_{net}^{(box)} = & -\frac{1}{2} \left(2\mu(1-\beta) - \left(\alpha \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(-y_i) + 1 \right) - \beta \right) \operatorname{erfc}(y_i) \right) \\
& \times \log \left(\frac{y_e e^{y_e^2} \operatorname{erfc}(-y_e) \left(\alpha \left(\sqrt{\pi} y_i e^{y_i^2} \operatorname{erfc}(-y_i) + 1 \right) - \beta \right)}{2\mu(1-\beta) \operatorname{erfc}(-y_e) y_i e^{y_i^2}} \right) \\
& -\frac{1}{2} \left(\left(\alpha \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(-y_i) + 1 \right) - \beta \right) \right) \log \left(\frac{\left(\alpha \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(-y_i) + 1 \right) - \beta \right) \operatorname{erfc}(y_i)}{2\mu(1-\beta)} \right) \\
& -\frac{1}{2} \left(\left(\alpha \left(-\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) + 1 \right) - \beta \right) \right) \log \left(\frac{\left(\alpha \left(-\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) + 1 \right) - \beta \right) \operatorname{erfc}(-y_i)}{2(1-\mu)(1-\beta)} \right) \\
& -\frac{1}{2} \left(2(1-\mu)(1-\beta) - \left(\alpha \left(-\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) + 1 \right) - \beta \right) \operatorname{erfc}(-y_i) \right) \\
& \times \log \left(\frac{y_e e^{y_e^2} \operatorname{erfc}(y_e) \left(\alpha \left(-\sqrt{\pi} y_i e^{y_i^2} \operatorname{erfc}(y_i) + 1 \right) - \beta \right)}{2(1-\mu)(1-\beta) \operatorname{erfc}(y_e) y_i e^{y_i^2}} \right) \\
& + \alpha y_i^2 - \alpha y_e^2 - (1-\beta) \log(2).
\end{aligned} \tag{203}$$

Continuing further we also obtain

$$\begin{aligned}
\Psi_{net}^{(box)} = & -\frac{1}{2} \left(2\mu(1-\beta) - \left(\alpha \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(-y_i) + 1 \right) - \beta \right) \operatorname{erfc}(y_i) \right) \log \left(\frac{y_e e^{y_e^2}}{y_i e^{y_i^2}} \right) \\
& -\frac{1}{2} \left(2(1-\mu)(1-\beta) - \left(\alpha \left(-\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) + 1 \right) - \beta \right) \operatorname{erfc}(-y_i) \right) \log \left(\frac{y_e e^{y_e^2}}{y_i e^{y_i^2}} \right)
\end{aligned}$$

$$\begin{aligned}
& -\frac{1}{2} \left(2\mu(1-\beta) - \left(\alpha \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(-y_i) + 1 \right) - \beta \right) \operatorname{erfc}(y_i) \right) \\
& \times \log \left(\frac{\alpha \left(\sqrt{\pi} y_i e^{y_i^2} \operatorname{erfc}(-y_i) + 1 \right) - \beta}{2\mu(1-\beta)} \right) \\
& -\frac{1}{2} \left(2(1-\mu)(1-\beta) - \left(\alpha \left(-\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) + 1 \right) - \beta \right) \operatorname{erfc}(-y_i) \right) \\
& \times \log \left(\frac{\alpha \left(-\sqrt{\pi} y_i e^{y_i^2} \operatorname{erfc}(y_i) + 1 \right) - \beta}{2(1-\mu)(1-\beta)} \right) \\
& -\frac{1}{2} \left(\alpha \left(\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(-y_i) + 1 \right) - \beta \right) \operatorname{erfc}(y_i) \log \left(\frac{\alpha \left(\sqrt{\pi} y_i e^{y_i^2} \operatorname{erfc}(-y_i) + 1 \right) - \beta}{2\mu(1-\beta)} \right) \\
& -\frac{1}{2} \left(\alpha \left(-\sqrt{\pi} e^{y_i^2} y_i \operatorname{erfc}(y_i) + 1 \right) - \beta \right) \operatorname{erfc}(-y_i) \log \left(\frac{\alpha \left(-\sqrt{\pi} y_i e^{y_i^2} \operatorname{erfc}(y_i) + 1 \right) - \beta}{2(1-\mu)(1-\beta)} \right) \\
& + \alpha y_i^2 - \alpha y_e^2 - (1-\beta) \log(2), \tag{204}
\end{aligned}$$

and

$$\begin{aligned}
\Psi_{net}^{(box)} &= -(1-\alpha) \log \left(\frac{y_e e^{y_e^2}}{y_i e^{y_i^2}} \right) + \mu(1-\beta) \log \left(\frac{2\mu(1-\beta)}{\alpha \left(\sqrt{\pi} y_i e^{y_i^2} \operatorname{erfc}(-y_i) + 1 \right) - \beta} \right) \\
&+ (1-\mu)(1-\beta) \log \left(\frac{2(1-\mu)(1-\beta)}{\alpha \left(-\sqrt{\pi} y_i e^{y_i^2} \operatorname{erfc}(y_i) + 1 \right) - \beta} \right) + \alpha y_i^2 - \alpha y_e^2 - (1-\beta) \log(2). \tag{205}
\end{aligned}$$

Finally, a couple of simple algebraic transformations give

$$\begin{aligned}
\Psi_{net}^{(box)} = I_{err}^{(box)}(\alpha, \beta) &= (\alpha - 1) \log \left(\frac{y_e}{y_i} \right) + \mu(1-\beta) \log \left(\frac{1-\beta}{\alpha \left(\sqrt{\pi} y_i e^{y_i^2} \operatorname{erfc}(-y_i) + 1 \right) - \beta} \right) \\
&+ (1-\mu)(1-\beta) \log \left(\frac{(1-\mu)(1-\beta)}{\alpha \left(-\sqrt{\pi} y_i e^{y_i^2} \operatorname{erfc}(y_i) + 1 \right) - \beta} \right) + y_i^2 - y_e^2 = I_{ldp}^{(box)}(\alpha, \beta). \tag{206}
\end{aligned}$$

It is not that hard to see from (157) and (206) that $I_{err}^{(box)} = I_{ldp}^{(box)} = I_{err,u}^{(bin,ub)}$ (of course $y_e \leftrightarrow y_1$ and $y_i \leftrightarrow y_2$) which ensures that the choice for ν , A_0 , c_3 , and γ made in (155) is indeed optimal. Moreover, in the lower tail regime ($\alpha < \alpha_w$, considerations from [15] ensure that one also has

$$\Psi_{net}^{(box)}(\alpha, \beta) = I_{cor}^{(box)}(\alpha, \beta) \triangleq \lim_{n \rightarrow \infty} \frac{\log P_{cor}^{(box)}}{n} = \Psi_{com}^{(box)} + \Psi_{int}^{(box)} + \Psi_{ext}^{(box)}, \tag{207}$$

where $\Psi_{com}^{(box)}$, $\Psi_{int}^{(box)}$, and $\Psi_{ext}^{(box)}$ are as in (170). As was the case for binary ℓ_1 in Section 2, what we presented above then automatically characterizes the box ℓ_1 's LDP. The following theorem summarizes what we presented above.

Theorem 15 (Box ℓ_1 's LDP). *Assume the setup of Theorem 12 and assume that a pair (α, β) is given. Let $P_{err}^{(box)}$ be the probability that the solutions of (2) and (118) coincide and let P_{cor} be the probability that the solutions of (2) and (118) do not coincide. Set*

$$f_1^{(box)}(y_2; \alpha, \beta, \mu) \triangleq \left(\frac{2\mu(1-\beta)y_2 e^{y_2^2}}{\alpha \left(\sqrt{\pi} y_2 e^{y_2^2} \operatorname{erfc}(-y_2) + 1 \right) - \beta} \right) - y_2 e^{y_2^2} \operatorname{erfc}(y_2). \tag{208}$$

Also let y_2 and y_1 satisfy the following **fundamental characterizations of the box ℓ_1 's LDP** and achieve the optimum in (171):

$$2\operatorname{erfinv}\left(\frac{f_1^{(box)}(y_2; \alpha, \beta, \mu) + f_1^{(box)}(-y_2; \alpha, \beta, 1 - \mu)}{f_1^{(box)}(y_2; \alpha, \beta, \mu) - f_1^{(box)}(-y_2; \alpha, \beta, 1 - \mu)}\right) \frac{e^{\operatorname{erfinv}\left(\frac{f_1^{(box)}(y_2; \alpha, \beta, \mu) + f_1^{(box)}(-y_2; \alpha, \beta, 1 - \mu)}{f_1^{(box)}(y_2; \alpha, \beta, \mu) - f_1^{(box)}(-y_2; \alpha, \beta, 1 - \mu)}\right)^2}}{f_1^{(box)}(y_2; \alpha, \beta, \mu) - f_1^{(box)}(-y_2; \alpha, \beta, 1 - \mu)} = 1. \quad (209)$$

and

$$y_1 = \operatorname{erfinv}\left(\frac{f_1^{(box)}(y_2; \alpha, \beta, \mu) + f_1^{(box)}(-y_2; \alpha, \beta, 1 - \mu)}{f_1^{(box)}(y_2; \alpha, \beta, \mu) - f_1^{(box)}(-y_2; \alpha, \beta, 1 - \mu)}\right). \quad (210)$$

Finally, let $I_{ldp}^{(box)}(\alpha, \beta)$ be defined through the following **box ℓ_1 's fundamental LDP rate function** characterization

$$\begin{aligned} I_{ldp}^{(box)}(\alpha, \beta) &\triangleq (\alpha - 1) \log\left(\frac{y_1}{y_2}\right) + \mu(1 - \beta) \log\left(\frac{1 - \beta}{\alpha(\sqrt{\pi}e^{y_2^2}\operatorname{erfc}(-y_2) + 1) - \beta}\right) \\ &\quad + (1 - \mu)(1 - \beta) \log\left(\frac{(1 - \mu)(1 - \beta)}{\alpha(-\sqrt{\pi}e^{y_2^2}\operatorname{erfc}(y_2) + 1) - \beta}\right) + y_2^2 - y_1^2. \end{aligned} \quad (211)$$

Then if $\alpha > \alpha_w$

$$I_{err}^{(box)}(\alpha, \beta) \triangleq \lim_{n \rightarrow \infty} \frac{\log P_{err}^{(box)}}{n} = I_{ldp}^{(box)}(\alpha, \beta). \quad (212)$$

Moreover, if $\alpha < \alpha_w$

$$I_{cor}^{(box)}(\alpha, \beta) \triangleq \lim_{n \rightarrow \infty} \frac{\log P_{cor}^{(box)}}{n} = I_{ldp}^{(box)}(\alpha, \beta). \quad (213)$$

Proof. Follows from the above discussion. \square

3.2.4 Phase transitions

In this section we will show how one can quickly determine the phase transitions for the box ℓ_1 utilizing Theorem 15 and the above considerations leading up to Theorem 15. We start by closely following what we presented in Section 2.3.6, and focus on those pairs (α, β) for which $y_1 = y_2$ in Theorem 15 (as mentioned in Section 2.3.6, we may not know a priori if such pairs do exist; however, the derivation below will confirm that such an assumption is actually correct). From (150) and (151) we have

$$\begin{aligned} y_2 e^{y_2^2} \operatorname{erfc}(-y_2) &= y_1 e^{y_1^2} \operatorname{erfc}(-y_1) = \left(\frac{2\mu(1 - \beta)y_2 e^{y_2^2}}{\alpha(\sqrt{\pi}y_2 e^{y_2^2} \operatorname{erfc}(-y_2) + 1) - \beta} \right) - y_2 e^{y_2^2} \operatorname{erfc}(y_2) = f_1^{(box)}(y_2; \alpha, \beta, \mu) \\ y_2 e^{y_2^2} \operatorname{erfc}(y_2) &= y_1 e^{y_1^2} \operatorname{erfc}(y_1) = \left(\frac{2(1 - \mu)(1 - \beta)y_2 e^{y_2^2}}{\alpha(-\sqrt{\pi}y_2 e^{y_2^2} \operatorname{erfc}(y_2) + 1) - \beta} \right) - y_2 e^{y_2^2} \operatorname{erfc}(-y_2) \\ &= -f_1^{(box)}(-y_2; \alpha, \beta, 1 - \mu). \end{aligned} \quad (214)$$

Transforming (214) a bit further we arrive at the following analogue of (110)

$$\begin{aligned} 1 &= \left(\frac{\mu(1-\beta)}{\alpha(\sqrt{\pi}y_2e^{y_2^2}\operatorname{erfc}(-y_2)+1)-\beta} \right) \\ 1 &= \left(\frac{(1-\mu)(1-\beta)}{\alpha(-\sqrt{\pi}y_2e^{y_2^2}\operatorname{erfc}(y_2)+1)-\beta} \right). \end{aligned} \quad (215)$$

From (215) we then easily have

$$\begin{aligned} \sqrt{\pi}y_2e^{y_2^2}\operatorname{erfc}(-y_2) &= \frac{\mu(1-\beta)+\beta}{\alpha} - 1 \\ \sqrt{\pi}y_2e^{y_2^2}\operatorname{erfc}(y_2) &= 1 + \frac{-\beta - (1-\mu)(1-\beta)}{\alpha}, \end{aligned} \quad (216)$$

and

$$\frac{1+\operatorname{erf}(y_2)}{1-\operatorname{erf}(y_2)} = \frac{\operatorname{erfc}(-y_2)}{\operatorname{erfc}(y_2)} = \frac{\mu(1-\beta)+\beta-\alpha}{\alpha-\beta-(1-\mu)(1-\beta)}. \quad (217)$$

From (217) we then determine $\operatorname{erf}(y_2)$ and y_2

$$\begin{aligned} \operatorname{erf}(y_2) &= \frac{\frac{\mu(1-\beta)+\beta-\alpha}{\alpha-\beta-(1-\mu)(1-\beta)} - 1}{\frac{\mu(1-\beta)+\beta-\alpha}{\alpha-\beta-(1-\mu)(1-\beta)} + 1} = \frac{1+\beta-2\alpha}{(2\mu-1)(1-\beta)} \\ \iff y_2 &= \operatorname{erfinv}\left(\frac{1+\beta-2\alpha}{(2\mu-1)(1-\beta)}\right). \end{aligned} \quad (218)$$

A combination of (216) and (218) then gives

$$\sqrt{\pi}y_2e^{y_2^2}\operatorname{erfc}(-y_2) = \sqrt{\pi}y_2e^{y_2^2}(1+\operatorname{erf}(y_2)) = \sqrt{\pi}y_2e^{y_2^2}\frac{2\mu(1-\beta)+2\beta-2\alpha}{(2\mu-1)(1-\beta)} = \frac{\mu(1-\beta)+\beta-\alpha}{\alpha}. \quad (219)$$

Finally combining (218) and (219) we obtain

$$\frac{(2\mu-1)(1-\beta)e^{-y_2^2}}{2\sqrt{\pi}\alpha y_2} = \frac{(2\mu-1)(1-\beta)e^{-(\operatorname{erfinv}(\frac{1+\beta-2\alpha}{(2\mu-1)(1-\beta)}))^2}}{2\sqrt{\pi}\alpha\operatorname{erfinv}\left(\frac{1+\beta-2\alpha}{(2\mu-1)(1-\beta)}\right)} = 1. \quad (220)$$

(220) effectively establishes the phase transition curve. Also, from (214) we have

$$f_1(y_2; \alpha, \beta) - f_1^{(box)}(-y_2; \alpha, \beta, 1-\mu) = y_2e^{y_2^2}\operatorname{erfc}(-y_2) + y_2e^{y_2^2}\operatorname{erfc}(y_2) = 2y_2e^{y_2^2}, \quad (221)$$

and

$$f_1(y_2; \alpha, \beta) + f_1(-y_2; \alpha, 1-\beta) = y_2e^{y_2^2}\operatorname{erfc}(-y_2) - y_2e^{y_2^2}\operatorname{erfc}(y_2) = 2y_2e^{y_2^2}\operatorname{erf}(y_2). \quad (222)$$

Assuming that the pair (α, β) is such that (220) holds one then also has $y_1 = y_2$ and (221) and (222) ensure that (209) and (210) hold and that $I_{ldp}^{(box)}(\alpha, \beta) = 0$ in (211) which is exactly the value that $I_{ldp}^{(box)}(\alpha, \beta)$ takes at the phase transition. The following theorem summarizes the above discussion and the obtained PT characterization. It is essentially a box ℓ_1 analogue to Theorem 1 which characterizes the binary ℓ_1 's PT.

Theorem 16. (*Exact box ℓ_1 's weak threshold/PT*) *Let A be an $m \times n$ matrix in (2) with i.i.d. standard normal components. Let μ be a real number such that $\mu \in [\frac{1}{2}, 1]$. Further, let the unknown \mathbf{x} in (2) be box-constrained k -sparse and let the locations of the elements of \mathbf{x} from $(0, 1)$ be arbitrarily chosen but fixed. Let k, m, n be large and let $\alpha_w = \frac{m}{n}$ and $\beta_w = \frac{k}{n}$ be constants independent of m and n . Further, let α_w and*

β_w satisfy the following **fundamental characterization of the box ℓ_1 's PT**

$$\xi_{\alpha_w, \mu}^{(box)}(\beta_w) \triangleq \psi_{\beta_w, \mu}^{(box)}(\alpha_w) \triangleq \frac{(2\mu-1)(1-\beta_w) \sqrt{\frac{1}{2\pi}} e^{-\left(\operatorname{erfinv}\left(\frac{1+\beta_w-2\alpha_w}{(2\mu-1)(1-\beta_w)}\right)\right)^2}}{\alpha_w \sqrt{2} \operatorname{erfinv}\left(\frac{1+\beta_w-2\alpha_w}{(2\mu-1)(1-\beta_w)}\right)} = 1. \quad (223)$$

Then:

1. If $\alpha > \alpha_w$ then with overwhelming probability the solution of (118) is the box-constrained k -sparse \mathbf{x} that solves (2).
2. If $\alpha < \alpha_w$ then with overwhelming probability the box-constrained k -sparse \mathbf{x} with given fixed locations of nonzero components is the solution of (2) and is **not** the solution of (118).

Proof. Follows from the above discussion. \square

We do mention that the above way of deriving the PT curve is presented as a box ℓ_1 analogue to what was done in Section 2.3.6. If one is interested solely in the phase transition and not necessarily in the LDPs the box ℓ_1 PT curve can be quickly derived using the methodology that we introduced in [14, 23, 25, 26]. Namely, looking at (125) one would simply need to determine for any α a $\beta \in (0, \alpha)$ such that

$$\lim_{n \rightarrow \infty} \frac{Ew(\mathbf{h}, S_w^{(box)})}{\sqrt{n}} = \min_{\nu \geq 0} \sqrt{E(\mu(1-\beta) \max(\mathbf{h}_i - \nu, 0)^2 + (1-\mu)(1-\beta) \max(\mathbf{h}_i + \nu, 0)^2 + \beta(\mathbf{h}_i + \nu)^2)} = \sqrt{\alpha}. \quad (224)$$

Following the standards that we set in [14, 23, 25, 26] this is now a fairly routine task and we leave it as an exercise to confirm that one indeed obtains that β satisfies the fundamental box ℓ_1 PT from (223).

The fundamental PT characterizations given in the above theorem are indeed well defined. Namely, for any given fixed $\alpha \in (0, 1)$ there will be a unique $\beta \in (0, \alpha)$ such that $\xi_{\alpha, \mu}^{(box)}(\beta) = 1$ and for any given fixed $\beta \in (0, 1)$ there will be a unique $\alpha \in (\beta, 1)$ such that $\psi_{\beta, \mu}^{(box)}(\alpha) = 1$. The arguments are similar to the ones that we presented in [17]. We leave their adaptation to the box ℓ_1 PTs given in the above theorem as an easy exercise as well. Finally, in Figure 4 we show the theoretical PT curves for the box ℓ_1 that one can obtain based on (223).

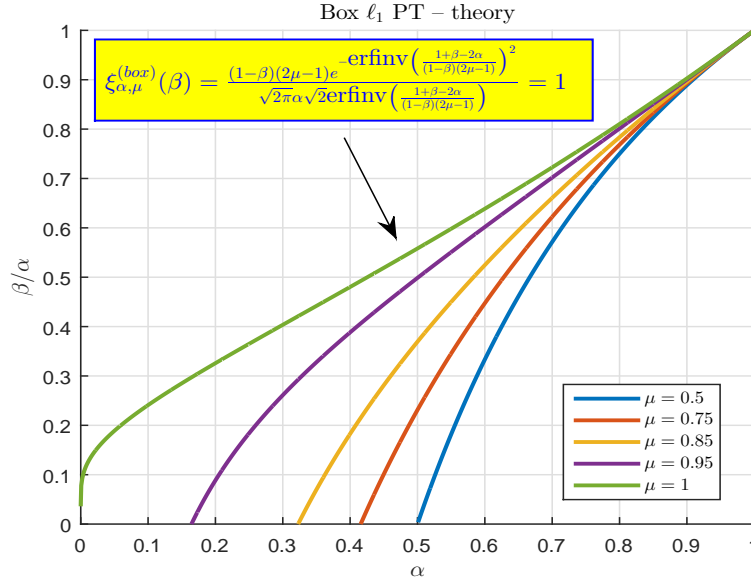


Figure 4: Box ℓ_1 's weak PT; $\{(\alpha, \beta) | \xi_{\alpha, \mu}^{(box)}(\beta) = 1\}$

3.3 Theoretical and numerical LDP results

In this section we finally give a little bit of a flavor as to what is actually proven in Theorem 15. These results are essentially box ℓ_1 analogues to the results presented in Section 2.4. Consequently, in presentation of the results, we try to maintain as much of a parallelism with Section 2.4 as possible. In Figure 5 we show the theoretical LDP rate function curve that one can obtain based on Theorem 15. This figure is complemented by Table 3 where we show the numerical values for all quantities of interest in Theorems 12 and 15 for several α 's from the transition zone (i.e. for several α 's around the breaking point; here $\beta = 0.18469$ is chosen such that the breaking point/threshold for $\alpha = 0.5$ and $\mu = 0.85$). Finally, in Figure 6 and Table 4 we show the comparison between the simulated values and the theoretical ones. As was the case for the binary ℓ_1 in Section 2.4, here we again observe that even for fairly small dimensions (of order 100) one already approaches the theoretical curves derived assuming an infinite dimensional asymptotic regime.

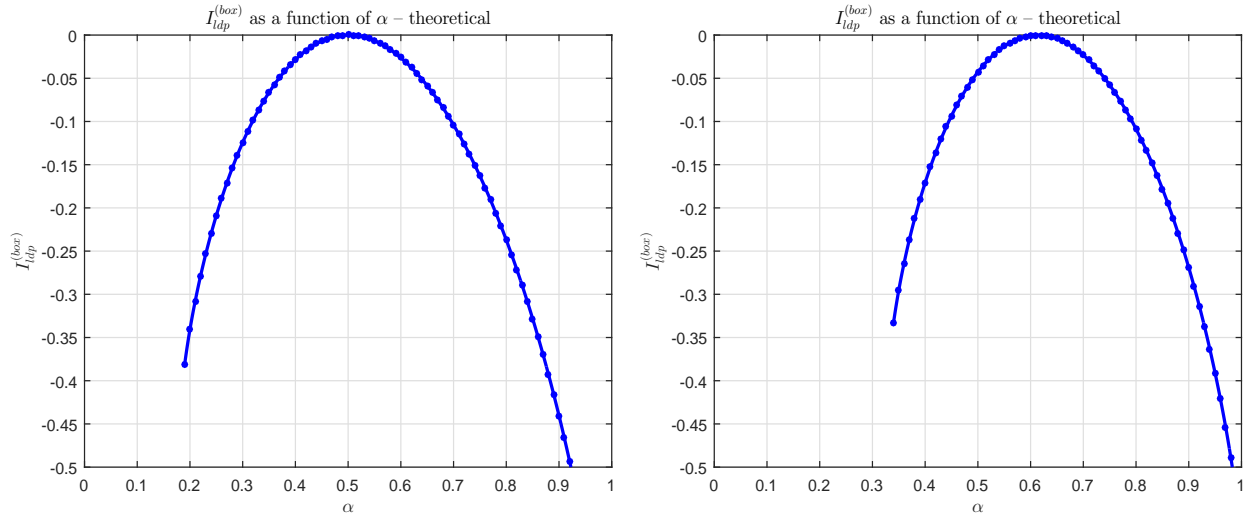


Figure 5: $I_{ldp}^{(box)}$ as a function of α for $\mu = 0.85$; left – $\beta = 0.18469$; right – $\beta = \frac{1}{3}$

Table 3: A collection of values for y_1 , γ_{box} , A_{box} , y_2 , ν , A_0 , c_3 , γ , and $I_{ldp}^{(box)}$ in Theorem 12; $\beta = 0.18469$, $\mu = 0.85$

α	0.40	0.45	0.50	0.55	0.60
y_1	0.3588	0.3270	0.2951	0.2635	0.2321
γ_{box}	0.0607	0.0504	0.0414	0.0336	0.0269
A_{box}	0.2806	0.2708	0.2612	0.2515	0.2418
y_2	0.1994	0.2473	0.2951	0.3429	0.3904
ν	0.5074	0.4624	0.4173	0.3726	0.3282
A_0	1.7994	1.3223	1.0000	0.7684	0.5945
c_3	-0.7866	-0.3797	0.0000	0.3952	0.8424
γ	0.1757	0.2537	0.3536	0.4825	0.6514
$I_{ldp}^{(box)}$	-0.0284	-0.0069	0.0000	-0.0066	-0.0262

4 Conclusion

This paper revisits the standard ℓ_1 heuristic and its a modification when used for solving random linear systems with structured sparse solutions. Two types of structuring on top of the standard sparsity are

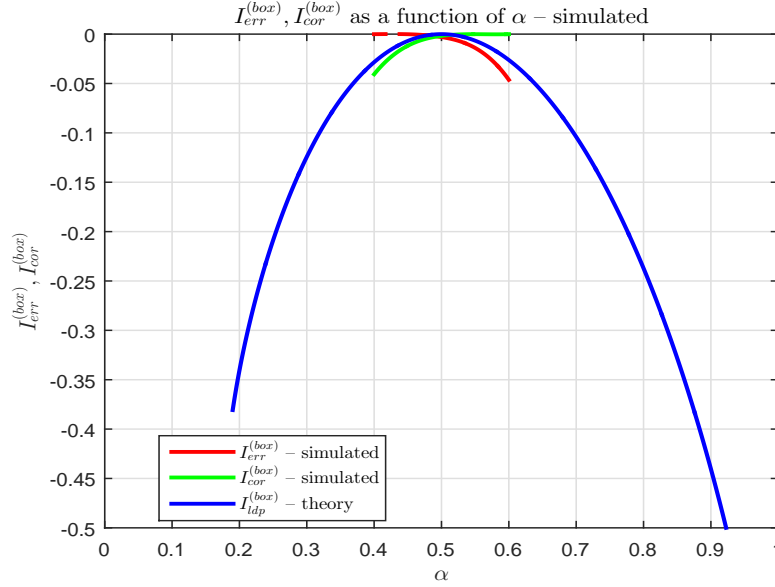


Figure 6: Box ℓ_1 's weak LDP rate function – theory and simulation; $\beta = 0.18469$, $\mu = 0.85$

Table 4: $I_{err}^{(box)}$, $I_{cor}^{(box)}$ – simulated; $I_{ldp}^{(box)}$ calculated for $\beta = 0.18469$ and $\mu = 0.85$

α	0.40	0.45	0.50	0.55	0.60
$\mu(n-k)$	87	208	208	208	87
k	23	55	55	55	23
m	50	135	150	165	75
n	125	300	300	300	125
$I_{err}^{(box)}$ – simulated	–0.0000	–0.0001	–0.0030	–0.0140	–0.0463
$I_{cor}^{(box)}$ – simulated	–0.0407	–0.0118	–0.0017	–0.0001	–0.0000
$I_{ldp}^{(box)}$ – theory	–0.0284	–0.0069	0.0000	–0.0066	–0.0262

considered here: 1) the solutions are binary, i.e. each component of the solution vector is from a set of only two values (these values are assumed to be a priori known); 2) the solutions are box-constrained, i.e. each component of the solution vector is either at one of the edges of the given interval or inside the interval. We looked at a relaxed modification of the standard ℓ_1 that we referred to as the binary or box ℓ_1 and how it fares when used for solving systems known to have solutions structured as above.

For both of these problems we presented the standard phase transition characterizations as well as a much deeper understanding of these phenomena by connecting them to the large deviations principles from the classical probability theory. A collection of very powerful probabilistic results that we obtained recently was often utilized. They turned out to be quite powerful even in the contexts of interest here and enabled us to explicitly characterize the large deviations in a manner similar to the one we showcased earlier when characterizing the phase transitions. Of particular importance in our view is that we were able to parallel the elegance we achieved earlier in the phase transitions characterizations in various other scenarios.

We also presented an alternative, high-dimensional geometry type of, view of the binary/box ℓ_1 . Through such an analysis we were again able to fully characterize the performance behavior of the modified ℓ_1 s. Consequently, we were able to show that the two substantially different analysis paths produce the same results (a conclusion certainly expected if the axioms of mathematics are properly set). To give a bit of a flavor as to what we actually proved in the paper, we also presented quite a few numerical results. They are in a very good agreement with all of our theoretical predictions (in fact, the simulated results indicate

that this theoretical/numerical agreement already happens for systems of rather small dimensions of order of few hundreds which is perhaps somewhat surprising given that the theoretical results, by the definitions of the LDPs, assume systems of very large, basically infinite, dimensions). Clearly, there are many ways that one can exploit to continue further and study various other aspects of the algorithms/problems at hand. A couple of cosmetic adjustments of the techniques introduced here and in a few of our earlier works are on occasion needed. However, the above mentioned conceptual elegance of the approaches that we presented ensures that these adjustments are now fairly routine tasks. Still, we will present some of them in several companion papers for a few related problems that we find interesting.

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